Measurement of the ratio of specific heats

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I. INTRODUCTION

The ratio of specific heats determines how hot a gas becomes when compressed adiabatically. In a monatomic gas, all mechanical work is transformed to molecular kinetic energy resulting in the highest temperatures and pressures when the gas in compressed compared to other gases. When the gas molecule has internal means to store energy (e.q. a diatomic), the temperatures and pressures are not as high as some energy is stored in rotational and vibrational modes. Due to the differing pressures, "springs" designed with gases of different specific heat ratios have different natural frequencies. In this paper we explore the design and operation of a device constructed with the purpose of inferring the specific heat ratio from measurement of the frequency response of a gas spring.

Using the frequency of mechanical oscillations to measure the ratio of specific heats has been known for some time. Ruchhardt proposed a method that used the oscillations of ball in the neck of a large bottle in 1929. Several modifications of this technique were developed including the method of Katz, Woods, and Leverton proposed in 1940. This technique used the frequency response of a forced piston in an enclosed tube and is the system we analyze in this paper [1] .

The basic system is shown in Figure 1. A piston of mass m is centered in a tube such that the gas on either side is contained in a cylinder of length L and cross sectional area A. An electromagnet outside the tube can apply a controlled oscillatory forcing and the natural frequency of the system can be observed.

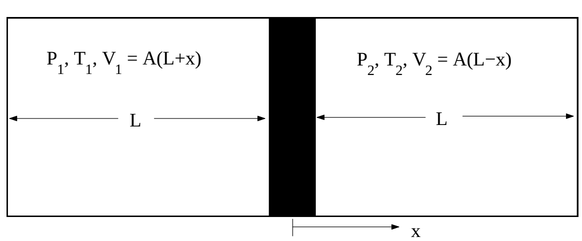

FIG. 1: Experimental setup. A piston of mass m is enclosed in a tube of cross sectional area A. The gas on each side of the piston is sealed. The pressure, temperature and volume of the gas are related by the ideal gas law.

II. FORMULATION

The motion of the piston is governed by Newton's Law and the fact that the only force we will consider is the pressure acting from opposite sides of the piston,

$$\frac{d^2x}{dt^2} = \frac{A}{m}(P_1(T_1, x) - P_2(T_1, x)). \tag{1}$$

where the pressure in gas 1 (2), $P_{1(2)}$, is a function of the temperature, $T_{1(2)}$, and the position of the piston, x. We consider a control mass of gas on either side of the piston and use the First Law of Thermodynamics to describe the evolution of the temperature in each chamber. The first law in rate form is

$$\frac{dU}{dt} = \dot{Q} - P\frac{dV}{dt} \tag{2}$$

where U is the total internal energy of the gas, $\dot{Q}$ is the rate of heat transfer (Watts), P is the pressure, and V is the volume. Assuming that the gas obeys the ideal gas law and that the specific heats are constant, the first law reduces to

$$\frac{dT}{dt} = \frac{\dot{Q}}{nC_v} - (\gamma - 1)\frac{T}{V}\frac{dV}{dt} \tag{3}$$

where n is the number of moles of gas in the chamber, C_v is the constant volume specific heat (J/mol · K), T is the temperature, and γ is the ratio of specific heats.

We assume that the walls of the tube are held at a constant temperature T_∞ and the primary mode of heat transfer is conduction through the gas and cylinder. While the rate of heat transfer depends upon the spatial and temporal evolution of the temperature field, we assume a simple model

$$\dot{Q} = -\frac{T - T_\infty}{R_{heat}}, \tag{4}$$

where R_{heat} is the total equivalent resistance to heat transfer These assumptions transform Equation 3 into

$$\frac{dT}{dt} = -\frac{T - T_\infty}{\tau} - (\gamma - 1)\frac{T}{V}\frac{dV}{dt} \tag{5}$$

where $\tau = R_{heat}nC_v$ is the time scale for heat transfer. The time scale τ can be interpreted physically as the time scale it takes for heat to diffuse out of the system.

Finally, we can non-dimensionalize the equations using the following scales: $[T] = T_\infty$, $[x] = L$, and $[t] = 1/\omega_o =$

[1] M. Zemansky, *Heat and Thermodynamics* (McGraw-Hill 1957).

$\sqrt{(Lm)/(2P_0A)}$, where P_0 is the initial pressure. This scheme results in the governing equations

$$\frac{d^2x}{dt^2} = \frac{1}{2}\left(\frac{T_1}{1+x} - \frac{T_2}{1-x}\right), \tag{6}$$

$$\frac{dT}{dt} = -\frac{T-1}{\tau\omega_o} - (\gamma-1)\frac{T}{V}\frac{dV}{dt}, \tag{7}$$

where $V = 1 \pm x$ for the left and right chambers of gas. This scaling is useful since there is only on free parameter in the system, $\tau\omega_o$.

When $\tau\omega_o << 1$ then the heat loss term is dominant and the system behaves isothermally. In this case the governing equations reduce to

$$\frac{d^2x}{dt^2} = \frac{1}{2}\left(\frac{1}{1+x} - \frac{1}{1-x}\right). \tag{8}$$

When the motions are very small the isothermal equation reduces to

$$\frac{d^2x}{dt^2} \approx -x. \tag{9}$$

Now we see that we can interpret our choice of time scale in the non-dimensionalization as the natural frequency of the spring system when the system is isothermal and motions are small. Therefore the comparison of time scales $\tau\omega_o << 1$ has the physical meaning: the time it takes heat to diffuse from the system is short compared to how long it takes the piston to undergo one cycle of oscillation.

When $\tau\omega_o >> 1$ the heat loss term is negligible and the system behaves adiabatically. Therefore the comparison of time scales $\tau\omega_o >> 1$ has the physical meaning: the time it takes heat to diffuse from the system is very long compared to how long it takes the piston to undergo one cycle of oscillation. When the heat loss term of equation 7 is removed the equation can be integrated analytically and the governing equations reduce to

$$\frac{d^2x}{dt^2} = \frac{1}{2}\left(\frac{1}{(1+x)^\gamma} - \frac{1}{(1-x)^\gamma}\right). \tag{10}$$

When the motions are very small the governing adiabatic equation becomes

$$\frac{d^2x}{dt^2} \approx -\gamma x. \tag{11}$$

In non-dimensional units we see that the system oscillates with a natural frequency $\sqrt{\gamma}$. The analysis implies that if we want the system to work (i.e. be able to measure the ratio of specific heats) then we only need to ensure that the single parameter $\tau\omega_o$ be large.

In all cases we will want to compute the entropy change in the universe. The entropy change of the gas at any instant is given by the ideal gas relation,

$$\frac{s}{R} = \frac{1}{\gamma-1}\ln\frac{T}{T_0} + \ln\frac{V}{V_0} \tag{12}$$

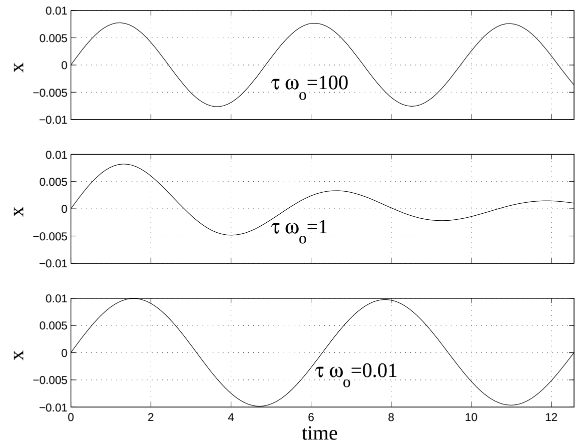

FIG. 2: Resulting oscillations for different initial value problems for different heat transfer time scales. The top figure is well insulated and approximates an adiabatic case, the middle figure has the natural frequency and heat transfer time scale equivalent, and the lower figure is the isothermal limit.

where T_0 and V_0 are the initial temperature and volume. Since we are using a non-dimensional formulation T_0 and V_0 are both unity.

The entropy change in the surroundings may be computed by integrating the rate of heat flow divided by the temperature of the surroundings. In dimensional units the intensive rate of change of entropy in the surroundings due to the heat flow is

$$\frac{ds_{sur}}{dt} = \frac{\dot{Q}}{nT_\infty}. \tag{13}$$

Substituting our heat transfer law and making use of our non-dimensional units this expression becomes

$$\frac{1}{R}\frac{ds_{sur}}{dt} = \frac{1}{(\gamma-1)}\left(\frac{T_1-1}{\omega_0\tau} + \frac{T_2-1}{\omega_0\tau}\right). \tag{14}$$

III. RESULTS

To test the behavior of the system we start by examining the oscillations history when the piston is started at $x = 0$ with a non-zero velocity. Figure 2 shows the position of the piston for three different values of $\tau\omega_o$. In the upper figure the heat takes a long time to diffuse from the system compared to the time it takes the system to undergo a cycle of oscillation (adiabatic). In this case the piston completes one oscillation with a period of $T = 2\pi/\sqrt{(\gamma)} = 4.87$ (the gas is assumed monatomic). In the lower figure the heat takes a short time to diffuse from the system compared to the time it takes the system to undergo a cycle of oscillation (isothermal). In this case the piston completes one oscillation with a period of $T = 2\pi$.

In both the isothermal and adiabatic limit the system oscillations continue with no damping. The oscillations

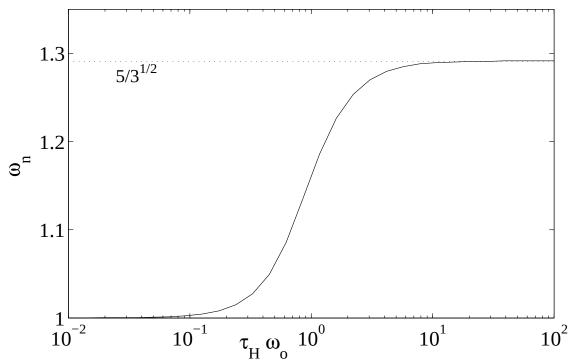

FIG. 3: Natural frequency of the system as a function of heat transfer time scale. Note that the equations are scaled by the natural frequency of the system when the system is isothermal, and the natural frequency is $\sqrt{\gamma}$ when the system is adiabatic. We can see from this figure that the system is nearly adiabatic when the parameter $\tau\omega_o > 10$.

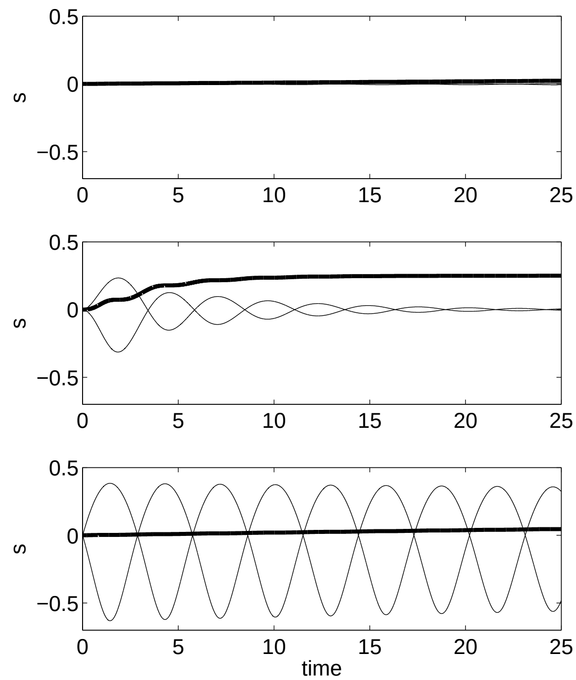

FIG. 4: Entropy of the gas in the two parts of the system and the total change entropy in the universe (thick line). The figures use $\tau\omega_o = 0.01$, $\tau\omega_o = 1$, $\tau\omega_o = 100$ from top to bottom. The top is nearly adiabatic, the middle has significant heat transfer, the bottom is nearly isothermal.

continue forever since the gas is undergoing an internally reversible process. There is no mechanism by which energy is dissipated on average from the system. In the case where the heat transfer occurs on the same time scale as the oscillations, the gas is undergoing an irreversible process and therefore the oscillations dampen.

We can look at how the natural frequency of the system changes as the system transitions from isothermal to adiabatic behavior. In Figure 3 we run the initial value problem for several different values of the $\tau\omega_o$. We find that the simulations predict a smooth transition from the isothermal to the adiabatic natural frequency. An important point regarding this figure is that the response is linear, i.e. we are considering a case where the initial velocity will induce motions only a small distance about the equilibrium.

In Figure 4 we show the entropy change in the two compartments of the gas and the entropy change in the universe as a function of time for three different cases. It is clear from the figures that our intuitive understandings are confirmed. In the nearly adiabtic case there is little entropy change in both the gas and the unverse. A slight drift upwards in the entropy of the universe is found due to the finite amount of heat transfer. In the case where the heat transfer time scale and the natural frequency are equivalent, we see that the entropy of the universe increases significantly. Entropy of the universe increases to a steady value until the system comes to rest. The total change in the universe's entropy after the piston comes to rest is related to the lost initial energy of the system. Finally, in the isothermal case we see that the entropy change of the two compartments of the gas is significant, however there is no significant change in the entropy of the universe. In the isothermal case, the heat pushed out on compression is pulled back into the system when the piston re-exands. There is no heat transfer across a finite temperature gradient and the isothermal process is reversible.

IV. CONCLUSIONS

The natural frequency of the gas spring system can be analyzed and simulated in a straightforward manner to provide the relationship between the frequency response and the ratio of specific heats. The use of a non-dimensional formulation reduces the system behavior a single parameter; the product of the natural frequency and the time scale for heat to flow out of the system. We found that the frequency does depend on the ratio of specific heats, but only when the system is acting in a nearly adiabatic manner. We confirmed our understanding of entropy by computing the entropy change of the system and universe and confirming that the adiabatic and isothermal limits are reversible processes.