Analysis of a draining tank

By Brian D. Storey

Franklin W. Olin College of Engineering

1. Equations

A tank of water is drained with siphon. The cylindrical tank has a constant cross sectional area, A, and volume, V. The draining tube has diameter, D, and length L. The end of the tube and the top of the water are separate in vertical distance by a height, $h(t)$. The water that leaves the tank flows down the tube providing the simple conservation of mass statement,

$$A\frac{dh}{dt} = -\frac{\pi D^2}{4}v, \tag{1.1}$$

where v is the velocity of the water in the tube. The velocity of the water in the tube is related to the height of the water through the modified Bernoulli's equation,

$$2gh = v^2(1 + f\frac{L}{D}), \tag{1.2}$$

where f is the friction factor. In the laminar flow regime ($Re < 2300$), the friction factor is given as,

$$f = \frac{64}{Re}, \tag{1.3}$$

where Re is the Reynolds number, $Re = vD/\nu$ and ν is the kinematic viscosity. For the laminar case we can solve directly for the velocity as a function of height using the quadratic equation,

$$v = \frac{-\frac{64\nu L}{D^2} + \sqrt{(\frac{64\nu L}{D^2})^2 + 8gh}}{2}. \tag{1.4}$$

2. Non-dimensionalization

We define the follow scales to make our equations dimensionless, $\hat{t} = t/t0$, $\hat{h} = h/H_0$, and $\hat{v} = v/v_0$. H_0 is the initial height of the top of the water relative to the exit of the tube, the other scales are arbitrary for now. Applying this scaling to the conservation of mass yields,

$$A\frac{d(H_0\hat{h})}{d(t_0\hat{t})} = -\frac{\pi D^2}{4}v_0\hat{v}, \tag{2.1}$$

which reduces to

$$\frac{d\hat{h}}{d\hat{t}} = -\frac{\pi D^2 t_o v_0}{AH_0 4}\hat{v}. \tag{2.2}$$

We can choose any scales we want so we set,

$$t_0 = \frac{AH_0 4}{\pi D^2 v_0}, \tag{2.3}$$

so that the conservation of mass equation reduces to

$$\frac{d\hat{h}}{d\hat{t}} = -\hat{v}. \tag{2.4}$$

For the velocity scale we set v_0 equal to the initial velocity coming from the tube when the height is $h = H_0$, i.e.

$$v_0 = \frac{-\frac{64\nu L}{D^2} + \sqrt{(\frac{64\nu L}{D^2})^2 + 8gH_0}}{2}. \tag{2.5}$$

Finally we need the dimensionless relationship between height and velocity. We return to the original Bernoulli expression which we can express in dimensionless numbers as,

$$2gH_0\hat{h} = v_0^2\hat{v}^2(1 + \frac{64}{Re_0}\frac{L}{D}\frac{1}{\hat{v}}), \tag{2.6}$$

or

$$\frac{2gH_0}{v_0^2}\hat{h} = \hat{v}^2(1 + \frac{64}{Re_0}\frac{L}{D}\frac{1}{\hat{v}}), \tag{2.7}$$

where $Re_0 = v_0 D/\nu$. If we return to the definition of v_0 (see equation 1.2 - not 2.5), we can easily show

$$\frac{2gH_0}{v_0^2} = (1 + \frac{64}{Re_0}\frac{L}{D}). \tag{2.8}$$

Therefore, Equation 2.7 reduces to

$$(1 + \frac{64}{Re_0}\frac{L}{D})\hat{h} = \hat{v}^2(1 + \frac{64}{Re_0}\frac{L}{D}\frac{1}{\hat{v}}), \tag{2.9}$$

Applying the quadratic formula, we obtain

$$\hat{v} = \frac{-\frac{64L}{Re_0 D} + \sqrt{(\frac{64L}{Re_0 D})^2 + 4(1 + \frac{64L}{Re_0 D})\hat{h}}}{2}. \tag{2.10}$$

3. Results

To summarize, the governing dimensionless equations are,

$$\frac{d\hat{h}}{d\hat{t}} = -\hat{v}, \tag{3.1}$$

$$\hat{v} = \frac{-\frac{64L}{Re_0 D} + \sqrt{(\frac{64L}{Re_0 D})^2 + 4(1 + \frac{64L}{Re_0 D})\hat{h}}}{2}. \tag{3.2}$$

There is only one free parameter is $\frac{64L}{Re_0 D}$.

It is instructive to think of two limits of this equation $L/D = 0$ and $L/D \to \infty$. In case of $L/D = 0$ it is easy to see the equations reduce to

$$\frac{d\hat{h}}{d\hat{t}} = -\sqrt{\hat{h}}. \tag{3.3}$$

The solution is found by integrating with the condition that $\hat{h}(\hat{t} = 0) = 1$,

$$\hat{h} = \left(1 - \frac{\hat{t}}{2}\right)^2. \tag{3.4}$$

The case where $L/D \to \infty$ is a little more subtle. When L/D is large the velocity-height relation can be approximated as

$$\hat{v} = \frac{-\frac{64L}{Re_0 D} + \sqrt{(\frac{64L}{Re_0 D})^2 + 4\frac{64L}{Re_0 D}\hat{h}}}{2}. \tag{3.5}$$

Using a Taylor series expansion on the square root term, this is further reduced (after a little work) to

$$\hat{v} = \hat{h} \tag{3.6}$$

such that the governing DE is

$$\frac{d\hat{h}}{d\hat{t}} = -\hat{h}. \tag{3.7}$$

The solution is found by integrating with the condition that $\hat{h}(\hat{t} = 0) = 1$,

$$\hat{h} = e^{-\hat{t}}. \tag{3.8}$$

The two solutions are plotted in figure 1. We find that when there is no friction, the draining of the tank follows Torricelli's law and drains in 2 units of dimensionless time. When the tube is long and the friction dominates, the tank never drains and the solution follows a classic exponential. All other solutions lie between these two limits and are determined by the value of $\frac{64L}{Re_0 D}$. These solutions are too difficult to determine analytically and can be solved numerically. Also note that the height of 1 is referenced to the total height between the top of the water and the exit of the tube. Therefore, depending on the height of the tank, the solution will lose physical meaning once $\hat{h}$ has dropped below the bottom of the tank.

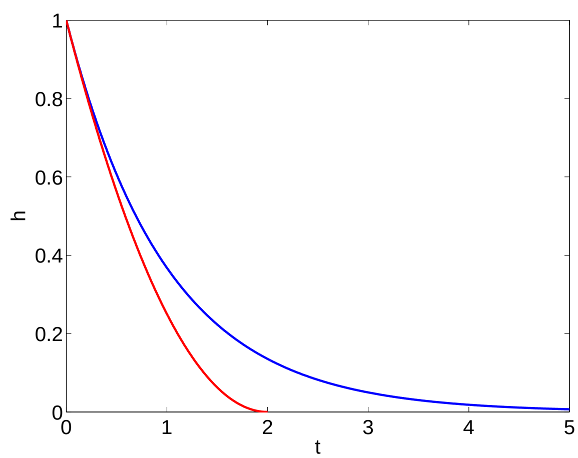

Figure 1. Solutions of the two limiting cases. With no friction the solution follows Torricelli's law. For the case dominated by friction the solution follows an exponential. All other solutions fall between these two cases and are determined by the value of $\frac{64L}{Re_0 D}$.