Signal Processing on the Permutahedron: Tight Spectral Frames for Ranked Data Analysis

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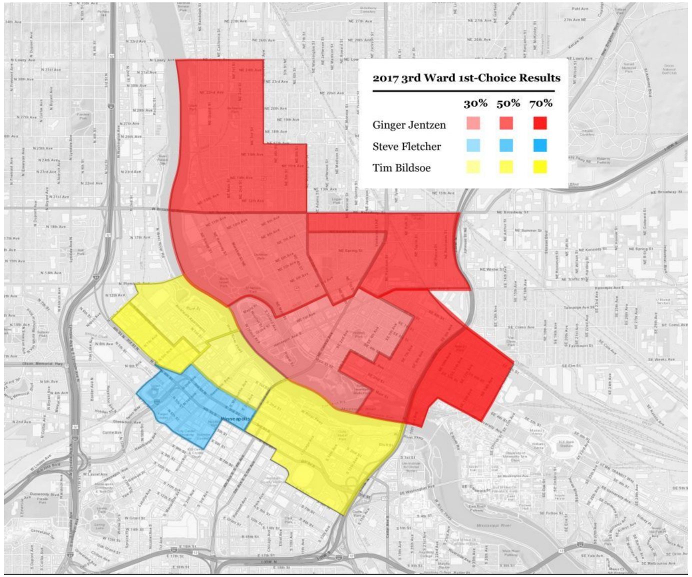
Ranked data example: 2017 Minneapolis City Council Ward 3 election

Four candidates:

1. Ginger Jentzen
   (Socialist-Alternative)
2. Samantha Pree-Stinson
   (Green)
3. Steven Fletcher
   (Democratic-Farmer-Labor, elected)
4. Tim Bildsoe
   (Democratic-Farmer-Labor)



source:
https://streets.mn/2019/05/20/how-the-2017-ward-3-election-in-minneapolis-foreshadows-our-local-political-future/

1st	2nd	3rd	4th	Count
4	3	2	1	574
4	3	1	2	201
4	2	3	1	131
4	2	1	3	32
4	1	3	2	89
4	1	2	3	46
3	4	2	1	422
3	4	1	2	271
3	2	4	1	159
3	2	1	4	243
3	1	4	2	156
3	1	2	4	204
2	4	3	1	111
2	4	1	3	30
2	3	4	1	161
2	3	1	4	145
2	1	4	3	56
2	1	3	4	153
1	4	3	2	255
1	4	2	3	77
1	3	4	2	376
1	3	2	4	421
1	2	4	3	204
1	2	3	4	538

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Ranked data lives on the permutahedron

1st	2nd	3rd	4th	Count
4	3	2	1	574
4	3	1	2	201
4	2	3	1	131
4	2	1	3	32
4	1	3	2	89
4	1	2	3	46
3	4	2	1	422
3	4	1	2	271
3	2	4	1	159
3	2	1	4	243
3	1	4	2	156
3	1	2	4	204
2	4	3	1	111
2	4	1	3	30
2	3	4	1	161
2	3	1	4	145
2	1	4	3	56
2	1	3	4	153
1	4	3	2	255
1	4	2	3	77
1	3	4	2	376
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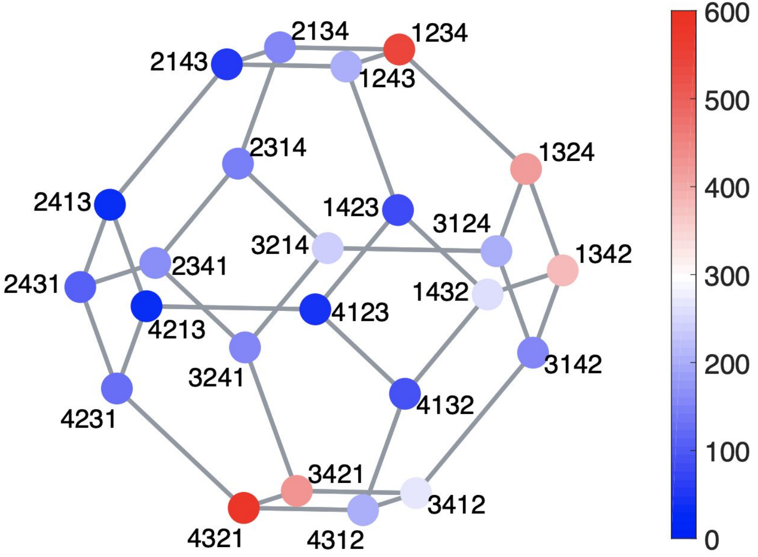

$$f : \mathbb{S}_n \to \mathbb{R}$$

- The permutahedron is the Cayley graph of the symmetric group generated by adjacent transpositions
- Rankings that differ by a single swap of neighboring candidates are close from a voter's viewpoint

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Applications and main research questions

- Applications
  - Political elections
  - Computer vision
  - Recommender systems
  - Bioinformatics

- Main research questions: How do we identify, interpret, and exploit structure in ranked data?

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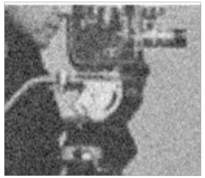
Data transforms

- Transforms and their inverses allow us to represent the same data in two different domains
- Potential benefits of mathematical transforms:
  - Easier/faster/more robust computations
    - e.g., polar coordinate transform for integration
  - New interpretations
    - e.g., representing a vector as a linear combination of eigenvectors for dynamical systems analysis
  - Structural patterns in the new coefficients can yield new data processing algorithms

How can we remove the noise from this image?



How can we compress the information in this signal down to just a few numbers?

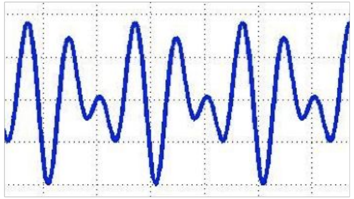
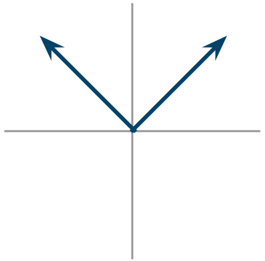
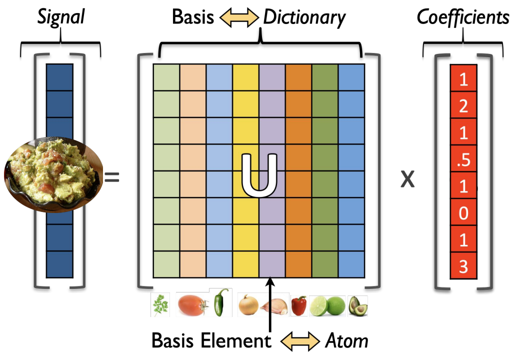
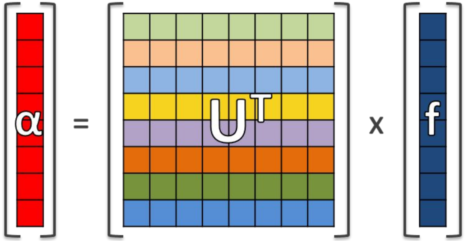
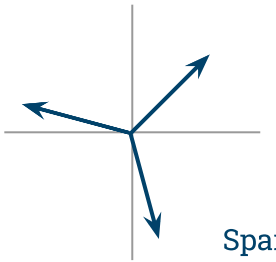
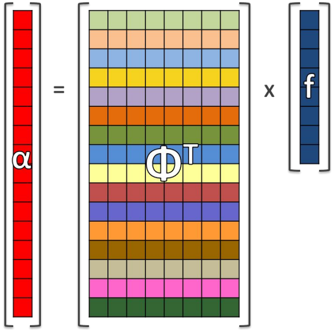


Image source: https://diy.dunnlumber.com/projects/how-to-build-a-picnic-table

See also:
http://tinyurl.com/wits-wavelets-starlet

Dictionaries

	Example in R^2	Synthesis	Analysis
Orthonormal basis			
Tight Parseval frame			

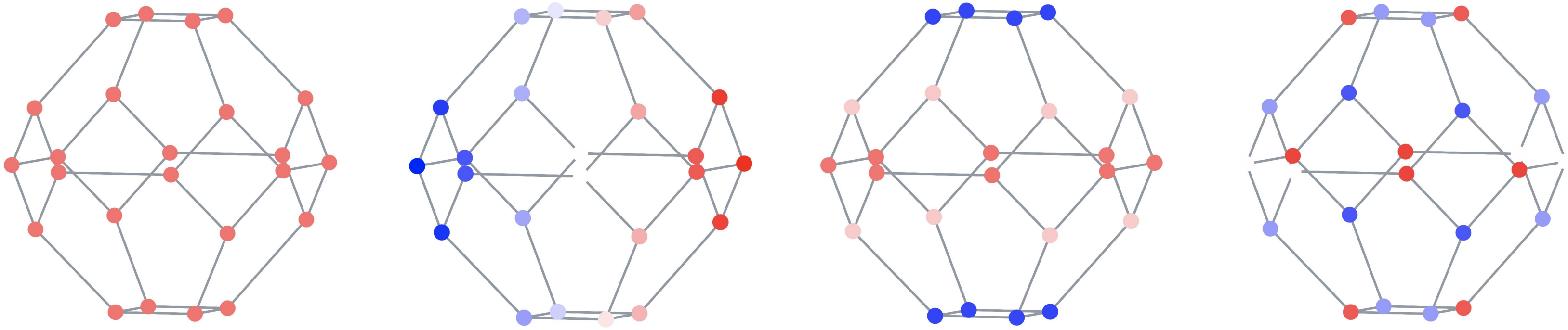
Sparse representations and/or interpretable atoms

- For finite dimensional spaces, any spanning set of vectors is a frame
- Shared properties of orthonormal bases and tight Parseval frames: (1) $\Phi\Phi^\top = I$, (2) $f = \sum_k \langle f, \varphi_k \rangle \varphi_k$ (3) $||f||^2 = ||\Phi^\top f||^2 = ||\alpha||^2$ (energy preservation)

Decompositions

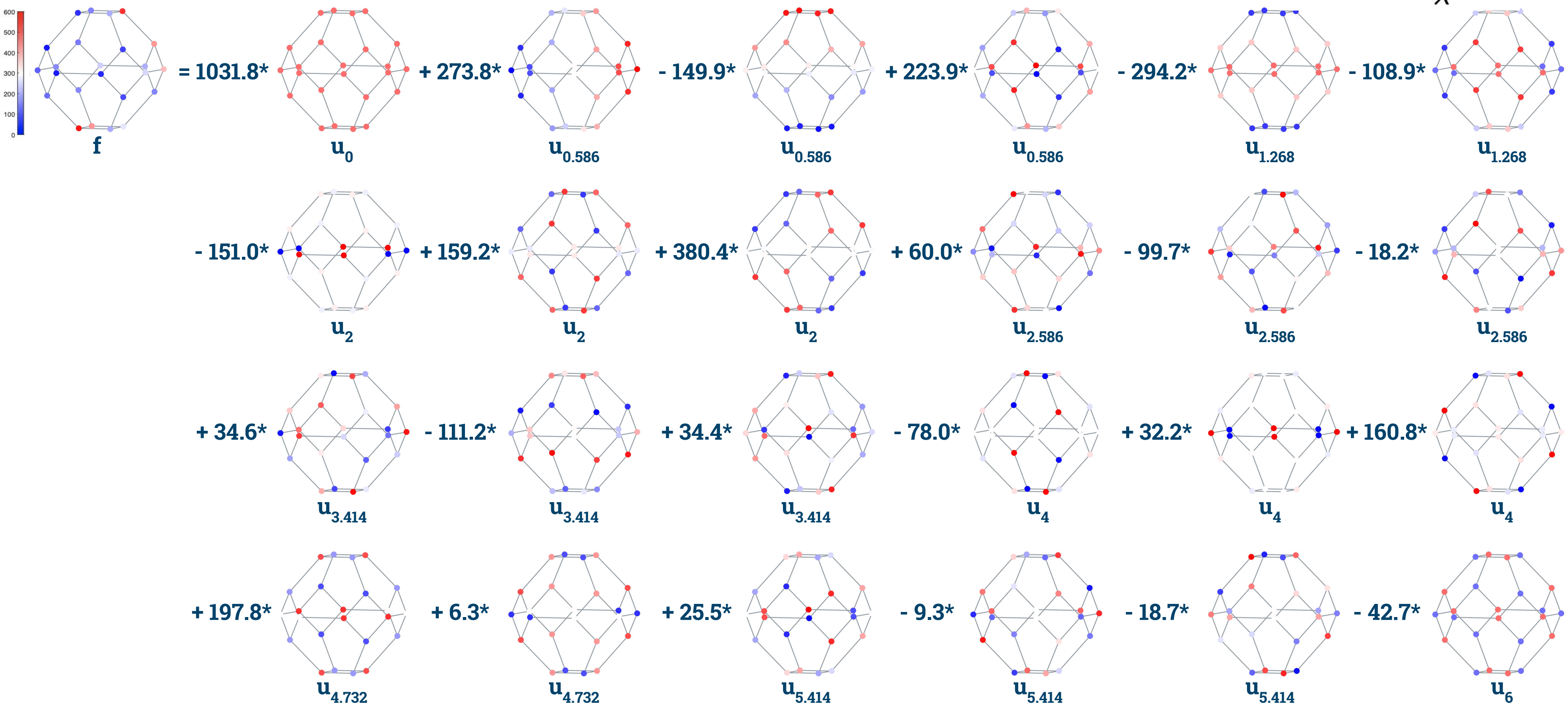
Graph signal processing approach: Spectral decomposition

- Graph Laplacian matrix: L=D-A
- Graph Laplacian eigenvectors are the analog of complex exponentials
- Values of the eigenvectors associated with low eigenvalues change less rapidly across connected vertices: $f^\top L f = \sum_{(i,j)\in\mathcal{E}} [f(i) - f(j)]^2$



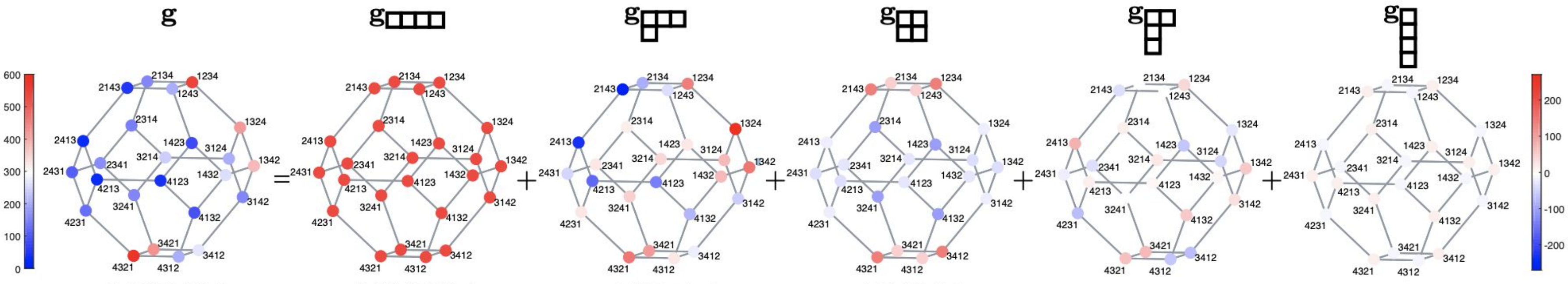
$\lambda = 0$ $\lambda = 0.586$ $\lambda = 1.268$ $\lambda = 4.732$

Graph signal processing approach:
Spectral decomposition / Graph Fourier transform $\mathbb{R}[\mathbb{S}_n] \cong \bigoplus_{\lambda} U_\lambda$



Group representation theory approach: Symmetry decomposition

- Represent the signal as the sum of projections onto each of the isotypic components
- $\mathbb{R}[\mathbb{S}_n] \cong \bigoplus_{\gamma \vdash n} W_\gamma$



1,607,249.0 = 1,064,709.4 + 355,201.6 + 137,575.3 + 47,942.6 + 1820.0

Energy decomposition

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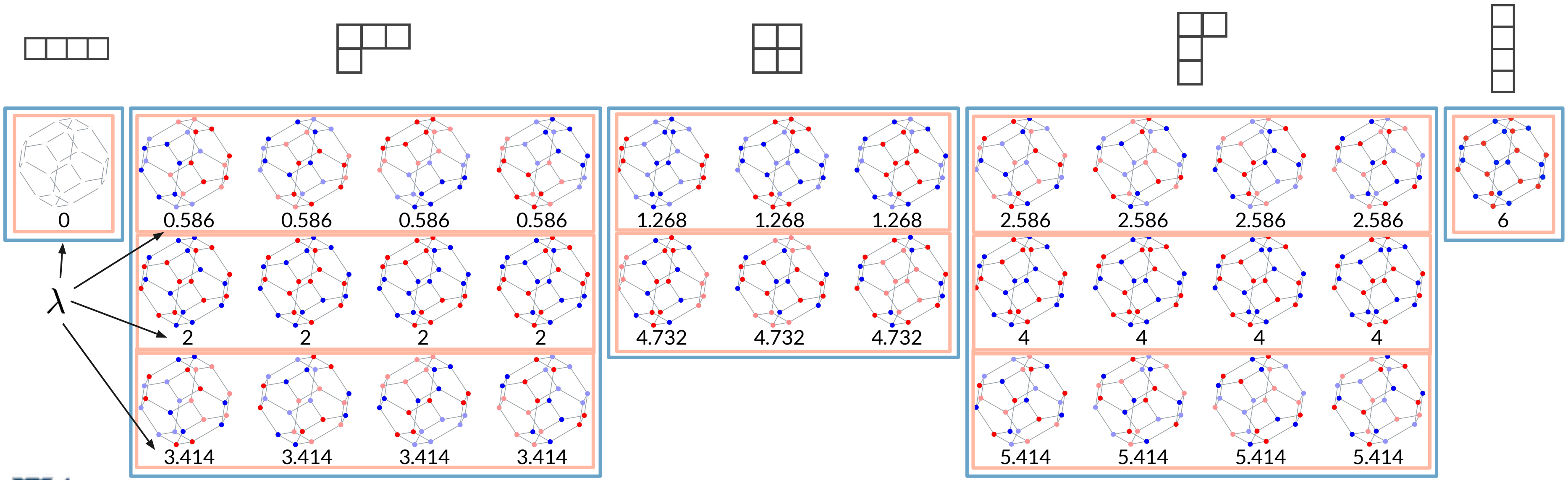
Our approach: Combine the spectral and symmetry decompositions

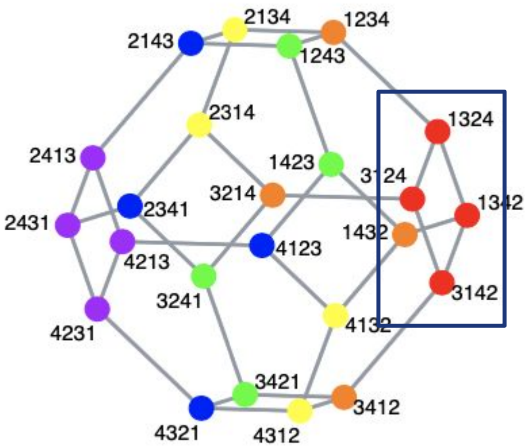
- $\mathbb{R}[\mathbb{S}_n] \cong \bigoplus_{\gamma \vdash n} \bigoplus_{\lambda \in \Lambda_\gamma} Z_{\gamma,\lambda}, \quad \text{where} \quad Z_{\gamma,\lambda} = W_\gamma \cap U_\lambda.$
- **Objective:** For each space $Z_{\gamma,\lambda}$, find a spanning set of dictionary atoms (vectors) with interpretable patterns that captures both **smoothness** and **structural** information of the ranked data on the permutahedron



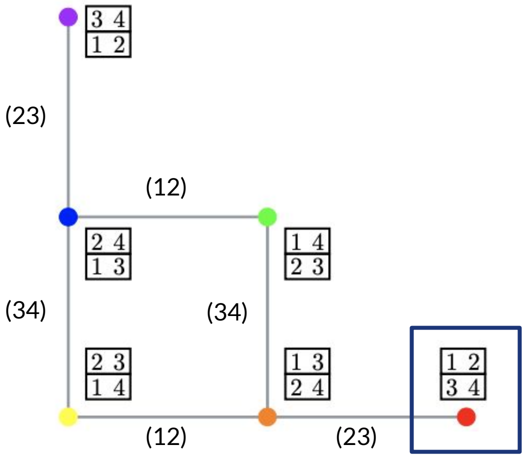
Tight Spectral Frames for Ranked Data

Background: Equitable partitions & Schreier graphs

Example equitable partition:
Group vertices (complete rankings) with candidates 1, 3 in the same ranking slots and candidates 2, 4 in the same ranking slots



Schreier graph: $\mathbb{P}_{[2,2]}$



	1 2 / 3 4	1 3 / 2 4	2 3 / 1 4	1 4 / 2 3	2 4 / 1 3	3 4 / 1 2
1234	·	1	·	·	·	·
1243	·	·	·	1	·	·
1324	1	·	·	·	·	·
1342	1	·	·	·	·	·
1423	·	·	·	1	·	·
1432	·	1	·	·	·	·
2134	·	·	1	·	·	·
2143	·	·	·	·	1	·
2314	·	·	1	·	·	·
2341	·	·	·	·	1	·
2413	·	·	·	·	·	1
2431	·	·	·	·	·	1
3124	1	·	·	·	·	·
3142	1	·	·	·	·	·
3214	·	1	·	·	·	·
3241	·	·	·	1	·	·
3412	·	1	·	·	·	·
3421	·	·	·	1	·	·
4123	·	·	·	·	1	·
4132	·	·	1	·	·	·
4213	·	·	·	·	·	1
4231	·	·	·	·	·	1
4312	·	·	1	·	·	·
4321	·	·	·	·	1	·

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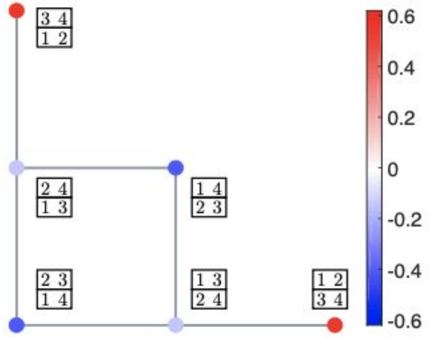
Tight frame construction

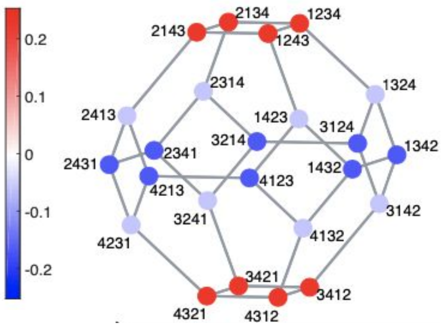
1. Compute a Laplacian eigenvector of a Schreier graph



$\gamma = [2, 2], \lambda = 1.2679$

2. Lift it to the permutahedron by assignment of candidates

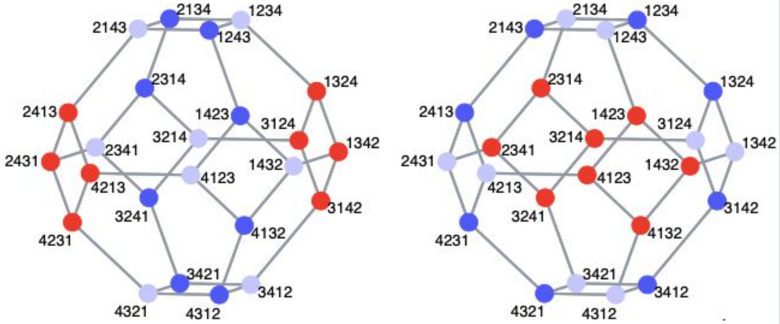
{12|34}



3. Rotate by group elements to obtain other frame vectors

{13|24}

{14|23}



Tight frame for $U_{1.2679}$

Note: We can also interpret each rotated frame vector as lifting by a different grouping of candidates

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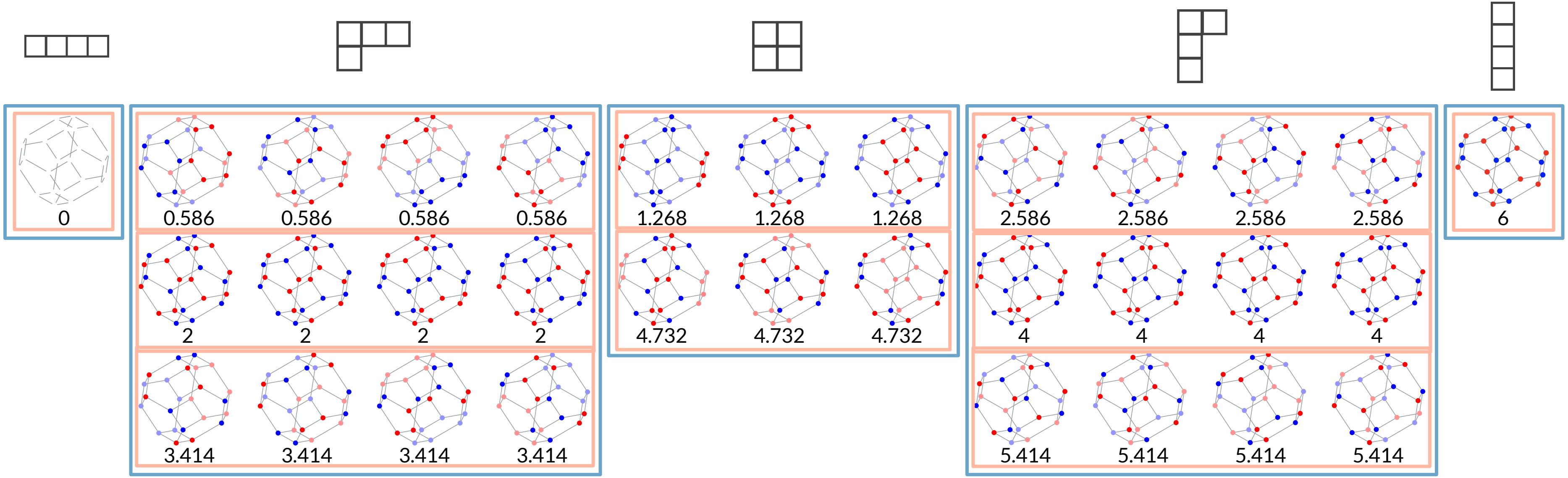
Example of a tight frame for $\mathbb{R}[S_4]$



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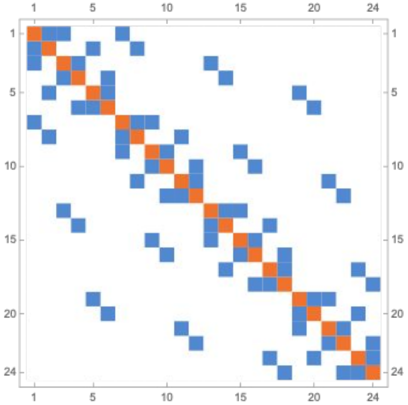
The Connection to Representation Theory

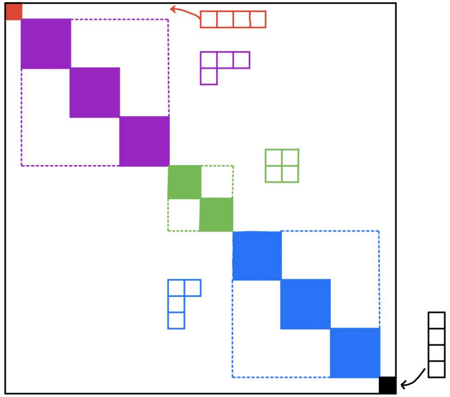
Representation Theory ↔ Spectral Graph Theory

- The graph **Laplacian** of $\mathbb{P}_n$ is the matrix of

$$\Phi_n = (n-1)\,\mathbb{1} - \sum_{i=1}^{n-1} (i, i+1) \quad \text{acting on } \mathbb{R}[S_n] \text{ on the right}$$



$\sim$



$$\mathbb{R}[S_n] \cong \bigoplus_{\gamma \vdash n} d_\gamma V_\gamma$$

- Laplacian eigenvalues fall into irreducible submodules (symmetry classes)

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Quotient Groups and Quotient Graphs

$\pi = \{1,5,7,9 \mid 3,4,8 \mid 2,6\}$ set partition of $\{1,\dots,n\}$

shape(π): $\gamma = [4,3,2] \vdash n$

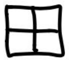
$\mathbb{R}[S_\pi \backslash S_n]$

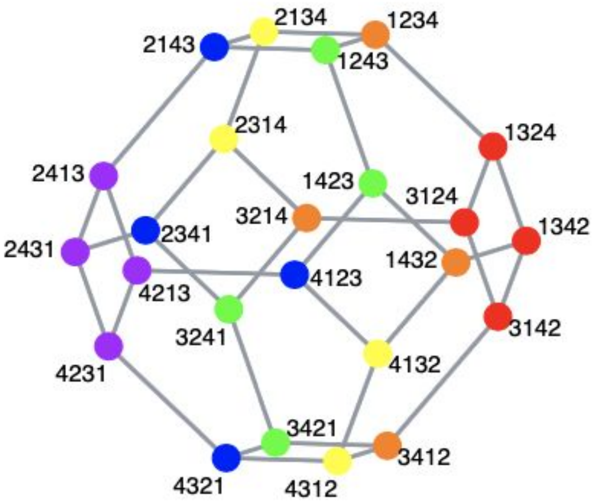
right coset representation

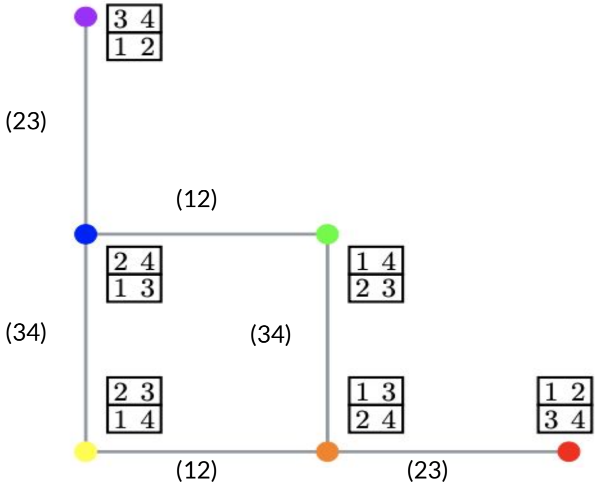
Young Subgroup: $S_\pi = S_{\{1,5,7,9\}} \times S_{\{3,4,8\}} \times S_{\{2,6\}}$

Example:

$\pi = \{1,3 \mid 2,4\}$

shape(π) = 





Frame Construction

		$V^*_{(6)}$	$V^*_{(5,1)}$	$V^*_{(4,2)}$	$V^*_{(4,1,1)}$	$V^*_{(3,3)}$	$V^*_{(3,2,1)}$
		1	5	9	10	5	16
$\mathbb{R}[\Pi_{(6)}]$	1	1					
$\mathbb{R}[\Pi_{(5,1)}]$	6	1	1				
$\mathbb{R}[\Pi_{(4,2)}]$	15	1	1	1			
$\mathbb{R}[\Pi_{(4,1,1)}]$	30	1	2	1	1		
$\mathbb{R}[\Pi_{(3,3)}]$	20	1	1	1	0	1	
$\mathbb{R}[\Pi_{(3,2,1)}]$	60	1	2	2	1	1	1

$\pi = \{1,5,7,9 \mid 3,4,8 \mid 2,6\}$ set partition of $\{1,\dots,n\}$

$M_\gamma \cong \mathbb{R}[S_\pi \backslash S_n]$ right coset representation

$\cong V_\gamma \oplus \bigoplus_{\nu \rhd \gamma} K_{\gamma,\nu} V_\nu$ Young's rule (Kostka numbers)

$v \leftarrow$ Laplacian eigenvector

$\{\{\sigma v_\pi \mid \sigma \in S_n\}\}$ Lift to $\mathbb{R}[S_n]$ (sum over cosets)

orbit under group action

Frame for V_γ in $\mathbb{R}[S_n]$

interpretability

$\sigma v_\pi = v_{\sigma(\pi)}$

Ranked Data Analysis:
Interpretation of the Analysis Coefficients

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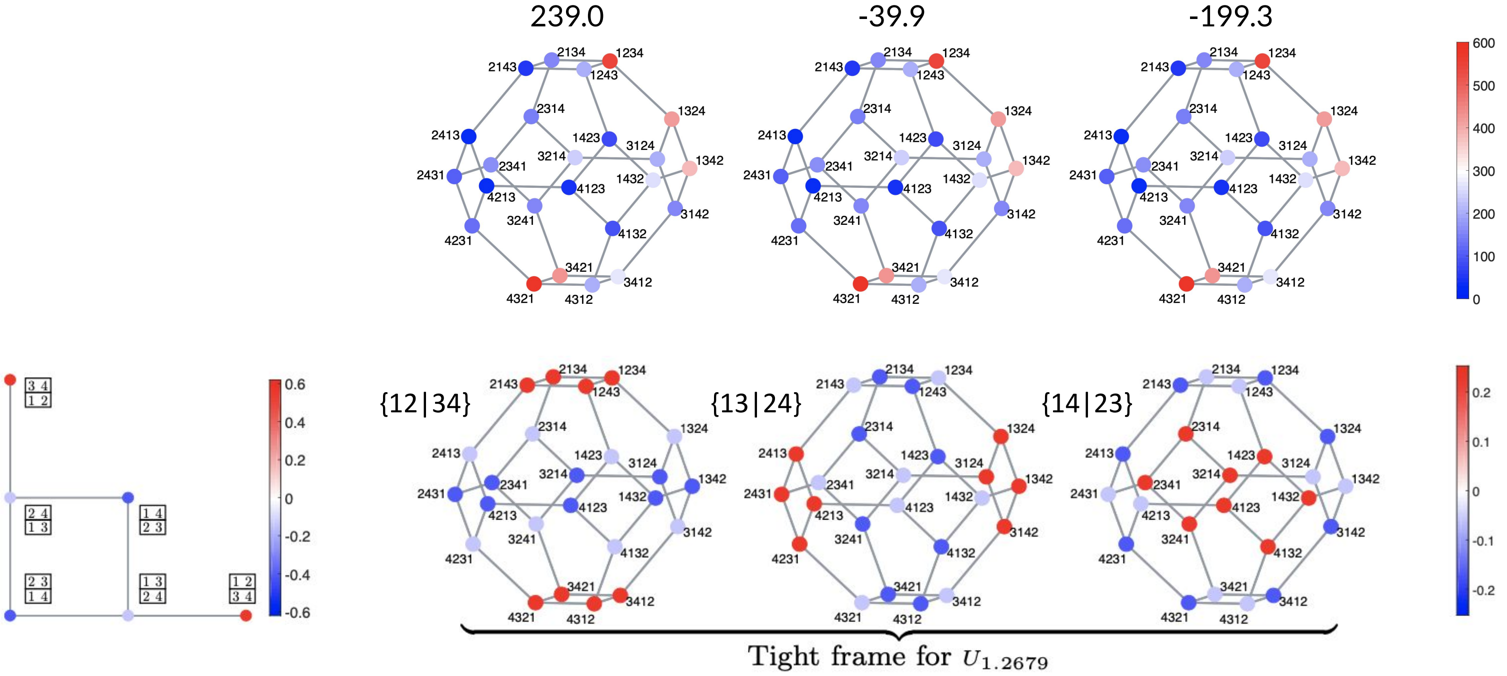
Candidate	First Choice	Second Choice	Third Choice	Fourth Choice
Ginger Jentzen	1871	704	922	1558
Samantha Pree-Stinson	656	1307	1744	1348
Steve Fletcher	1455	1878	1277	445
Tim Bildsoe	1073	1166	1112	1704

Analysis coefficients: Inner products between the signal on the permutahedron and each frame vector

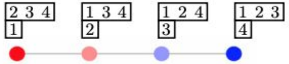
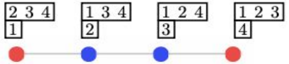
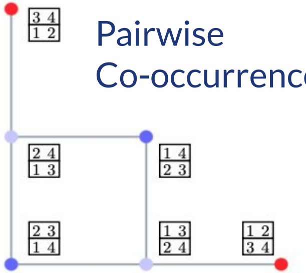
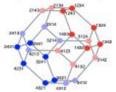
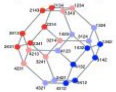
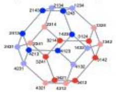
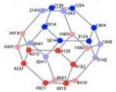
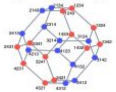
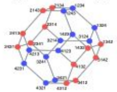
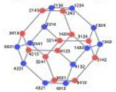
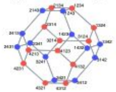
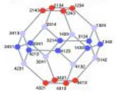
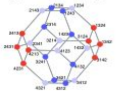
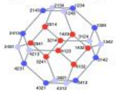


Signal (shown three times)

Frame vectors

Interpretation of analysis coefficients

γ										
λ	0	0.586	2	3.414	1.268	4.732	2.586	4	5.414	6
$\sum_{\bar\pi} \lvert\langle \mathbf{g}, \varphi_{\gamma,\lambda,\bar\pi}\rangle\rvert^2$	1064709.4	147617.5	192845.1	14739.0	98412.8	39162.5	13878.0	32979.6	1085.0	1820.0

γ											
λ	0.586				2				1.268		
$\mathbf{v}_\lambda$	Individual Popularity  • Positive: popular • Negative: unpopular				Polarization  • Positive: polarized • Negative: ranked middle				 Pairwise Co-occurrence		
$\bar\pi$	{234|1}	{134|2}	{124|3}	{123|4}	{234|1}	{134|2}	{124|3}	{123|4}	{12|34}	{13|24}	{14|23}
$\varphi_{\gamma,\lambda,\bar\pi}$											
$\langle \mathbf{g}, \varphi_{\gamma,\lambda,\bar\pi}\rangle$	51.4	-201.6	290.8	-140.6	318.7	-185.1	-221.9	88.2	239.0	-39.9	-199.3

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Sushi preference data (n=10)

Index	Sushi Type
1	Shrimp
2	Sea eel
3	Tuna
4	Squid
5	Sea urchin
6	Salmon roe
7	Egg
8	Fatty tuna
9	Tuna roll
10 (0)	Cucumber roll

- n=10: 10!=3.6 million permutations, 25.2 million frame vectors, ...
- This necessitated more efficient computation which drove interesting theoretical questions

1. Recursively build permutahedron/eigenvectors
2. Work in lower dimensional spaces when possible (do all computations on Schreiers)
3. Rotate data instead of using different projection matrices

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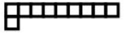
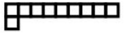
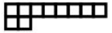
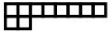
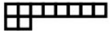
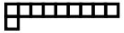
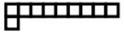
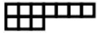
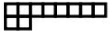
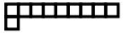
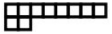
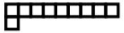
Analysis coefficients with the largest magnitudes

γ	$\bar{\pi}$	λ	$\langle \mathbf{h}, \boldsymbol{\varphi}_{\gamma,\lambda,\bar{\pi}} \rangle$	$\lvert \langle \mathbf{h}, \boldsymbol{\varphi}_{\gamma,\lambda,\bar{\pi}} \rangle \rvert^2$
	{1234567890}	0	2.6248	6.8893
	{123456789|0}	0.0979	-2.1513	4.6280
	{123456790|8}	0.0979	1.9978	3.9912
	{12345679|80}	0.2047	-1.7150	2.9413
	{12345689|70}	0.2047	1.6543	2.7369
	{12345679|8|0}	0.4799	1.3471	1.8147
	{12456790|38}	0.2047	1.3304	1.7699
	{123456890|7}	0.0979	-1.1896	1.4150
	{123456780|9}	0.3820	-1.1006	1.2112
	{1234569|780}	0.3227	1.0659	1.1362
	{12345690|78}	0.2047	-1.0400	1.0817
	{123456790|8}	0.3820	1.0392	1.0800
	{12345689|70}	0.4700	-1.0046	1.0093
	{123467890|5}	0.3820	0.9604	0.9223

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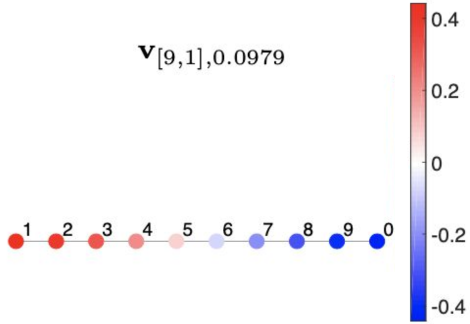
Interpretation of analysis coefficients

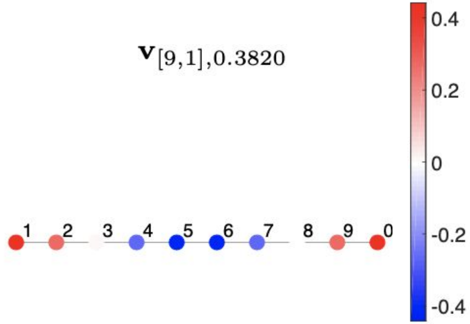
γ	$\bar{\pi}$	λ	$\langle \mathbf{h}, \boldsymbol{\varphi}_{\gamma,\lambda,\bar{\pi}} \rangle$	$\lvert\langle \mathbf{h}, \boldsymbol{\varphi}_{\gamma,\lambda,\bar{\pi}} \rangle\rvert^2$
[10]	{1234567890}	0	2.6248	6.8893
[9,1]	{123456789|0}	0.0979	-2.1513	4.6280
[9,1]	{123456790|8}	0.0979	1.9978	3.9912
[8,2]	{12345679|80}	0.2047	-1.7150	2.9413
[8,2]	{12345689|70}	0.2047	1.6543	2.7369
[8,1,1]	{12345679|8|0}	0.4799	1.3471	1.8147
[8,2]	{12456790|38}	0.2047	1.3304	1.7699
[9,1]	{123456890|7}	0.0979	-1.1896	1.4150
[9,1]	{123456780|9}	0.3820	-1.1006	1.2112
[7,3]	{1234569|780}	0.3227	1.0659	1.1362
[8,2]	{12345690|78}	0.2047	-1.0400	1.0817
[9,1]	{123456790|8}	0.3820	1.0392	1.0800
[8,2]	{12345689|70}	0.4700	-1.0046	1.0093
[9,1]	{123467890|5}	0.3820	0.9604	0.9223



Candidate	Coefficient
10 (Cucumber)	-2.15
8 (Fatty Tuna)	1.99
7 (Egg)	-1.19

Individual Popularity

- Positive: popular
- Negative: unpopular



Candidate	Coefficient
9 (Tuna Roll)	-1.1006
8 (Fatty Tuna)	1.0392
5 (Sea Urchin)	0.9604

Polarization

- Positive: polarized
- Negative: ranked middle

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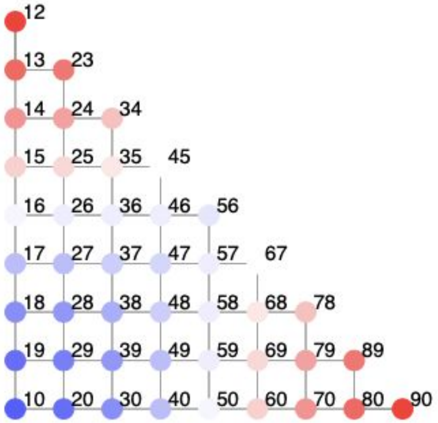
Interpretation of analysis coefficients

γ	$\bar{\pi}$	λ	$\langle \mathbf{h}, \boldsymbol{\varphi}_{\gamma,\lambda,\bar{\pi}} \rangle$	$\lvert\langle \mathbf{h}, \boldsymbol{\varphi}_{\gamma,\lambda,\bar{\pi}} \rangle\rvert^2$
[10]	{1234567890}	0	2.6248	6.8893
[9,1]	{123456789|0}	0.0979	-2.1513	4.6280
[9,1]	{123456790|8}	0.0979	1.9978	3.9912
[8,2]	{12345679|80}	0.2047	-1.7150	2.9413
[8,2]	{12345689|70}	0.2047	1.6543	2.7369
[8,1,1]	{12345679|8|0}	0.4799	1.3471	1.8147
[8,2]	{12456790|38}	0.2047	1.3304	1.7699
[9,1]	{123456890|7}	0.0979	-1.1896	1.4150
[9,1]	{123456780|9}	0.3820	-1.1006	1.2112
[7,3]	{1234569|780}	0.3227	1.0659	1.1362
[8,2]	{12345690|78}	0.2047	-1.0400	1.0817
[9,1]	{123456790|8}	0.3820	1.0392	1.0800
[8,2]	{12345689|70}	0.4700	-1.0046	1.0093
[9,1]	{123467890|5}	0.3820	0.9604	0.9223

$\mathbf{v}_{[8,2],0.2047}$



Candidates	Coefficient
8 (Fatty Tuna), 10 (Cucumber)	-1.7150
7 (Egg), 10 (Cucumber)	1.6543
3 (Tuna), 8 (Fatty Tuna)	1.3304
7 (Egg), 8 (Fatty Tuna)	-1.0400

Pairwise Co-occurrence

- Positive: ranked together
- Negative: ranked far apart

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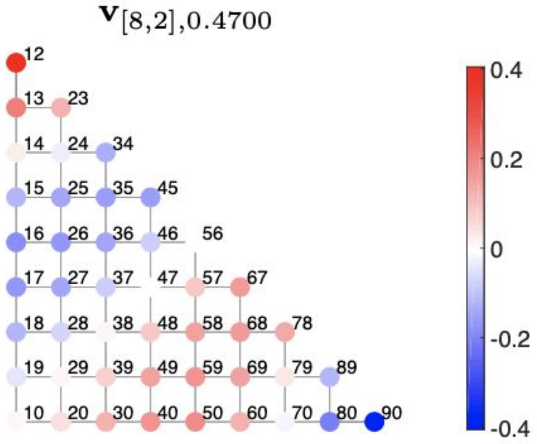
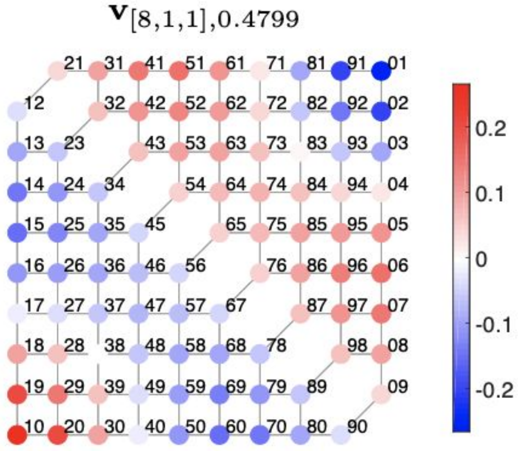
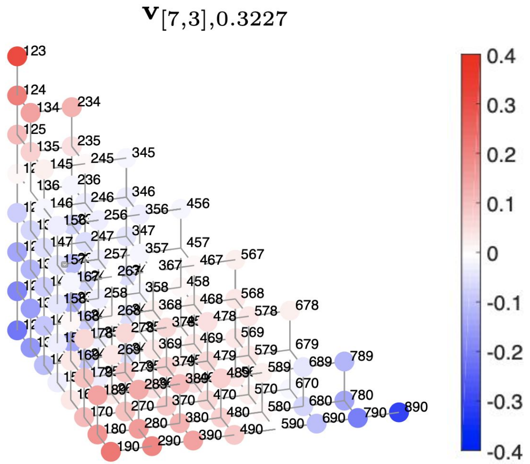
Interpretation of analysis coefficients

γ	$\bar{\pi}$	λ	$\langle \mathbf{h}, \boldsymbol{\varphi}_{\gamma,\lambda,\bar{\pi}} \rangle$	$\lvert\langle \mathbf{h}, \boldsymbol{\varphi}_{\gamma,\lambda,\bar{\pi}} \rangle\rvert^2$
	{1234567890}	0	2.6248	6.8893
	{123456789|0}	0.0979	-2.1513	4.6280
	{123456790|8}	0.0979	1.9978	3.9912
	{12345679|80}	0.2047	-1.7150	2.9413
	{12345689|70}	0.2047	1.6543	2.7369
	{12345679|8|0}	0.4799	1.3471	1.8147
	{12456790|38}	0.2047	1.3304	1.7699
	{123456890|7}	0.0979	-1.1896	1.4150
	{123456780|9}	0.3820	-1.1006	1.2112
	{1234569|780}	0.3227	1.0659	1.1362
	{12345690|78}	0.2047	-1.0400	1.0817
	{123456790|8}	0.3820	1.0392	1.0800
	{12345689|70}	0.4700	-1.0046	1.0093
	{123467890|5}	0.3820	0.9604	0.9223







Ongoing Work and Photographic Evidence

Ongoing work

- Generalization of the tight spectral frame construction to other finite groups and combinatorial structures
- Extension to partial ranking (ties allowed) and incomplete rankings (voters rank a subset of the candidates)
- More signal processing concepts on the permutahedron: wavelets, uncertainty principles

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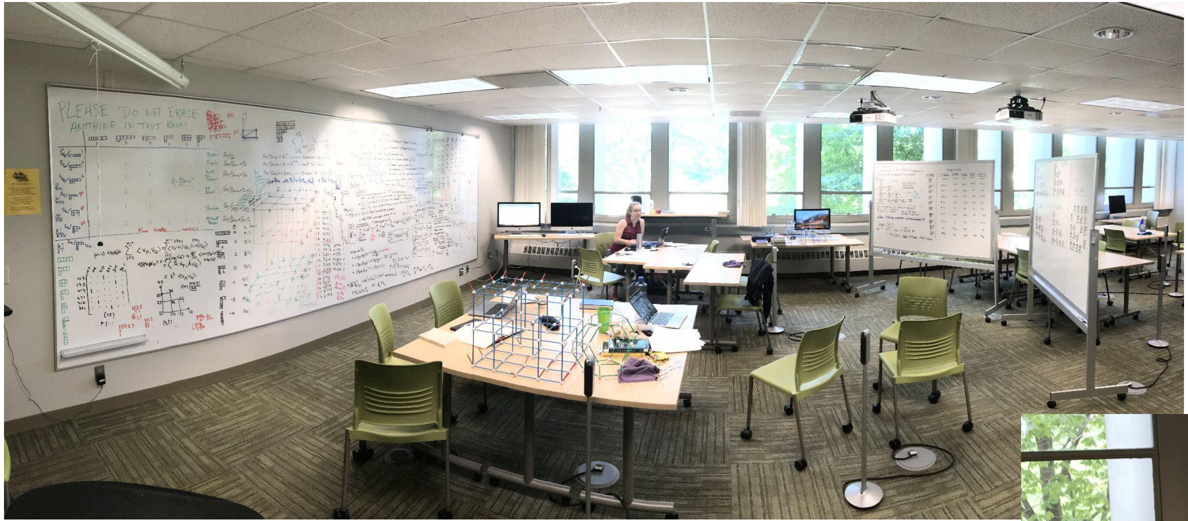
PLEASE DO NOT ERASE
ANYTHING IN THIS ROOM