A Scalable M-Channel Critically Sampled Graph Filter Bank

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Joint work with Shuni Li and Yan Jin

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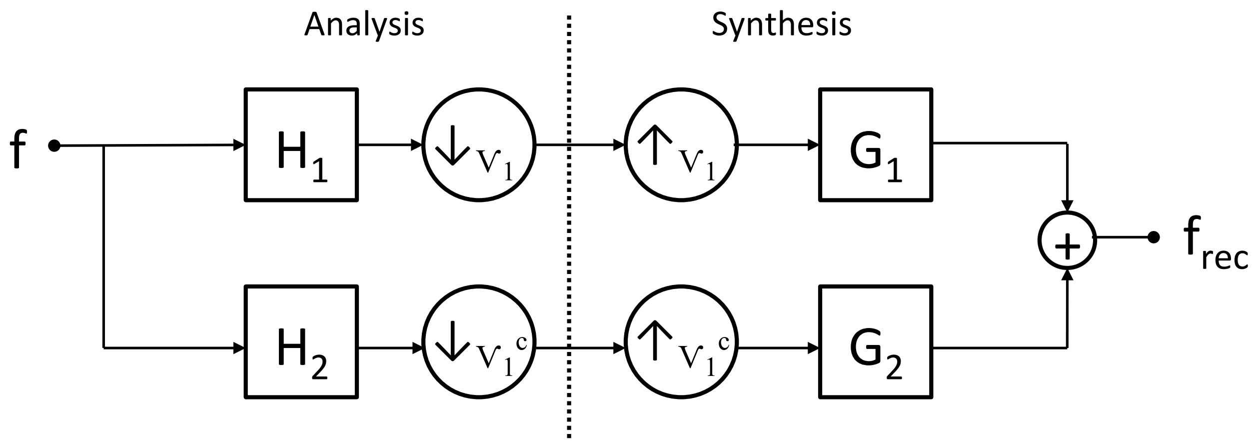
Graph Signal Processing Workshop

Lausanne, Switzerland

MACALESTER

Graph Spectral Filter Banks

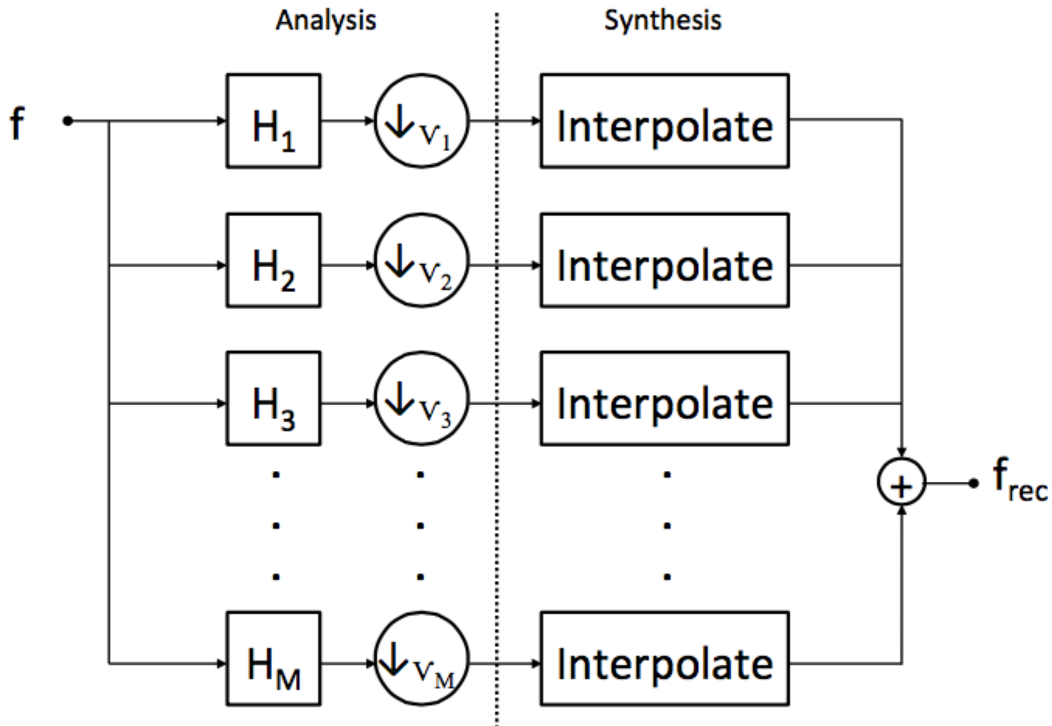


- For irregular graphs, it is difficult to generalize conditions on filters ensuring properties such as perfect reconstruction, orthogonality
- Approach 1: Decompose into structured subgraphs

  - Narang and Ortega, "Perfect reconstruction two-channel wavelet filter banks for graph structured data," TSP, 2012
  - Enkambaram et al., "Critically-sampled perfect reconstruction spline-wavelet filterbanks for graph signals," GlobalSIP, 2013
  - Kotzagiannidis and Dragotti, "The graph FRI framework - spline wavelet theory and sampling on circulant graphs," ICASSP, 2016

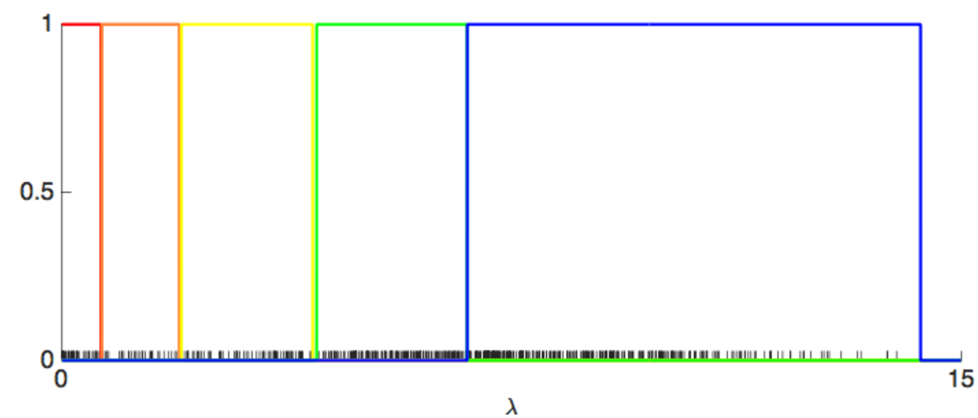
Approach 2: Replace Upsampling and Synthesis Filters with Interpolation Operators

Architecture



- Number of vertices in V_m is equal to the number of eigenvalues in the support of the corresponding filter

Ideal Filter Bank

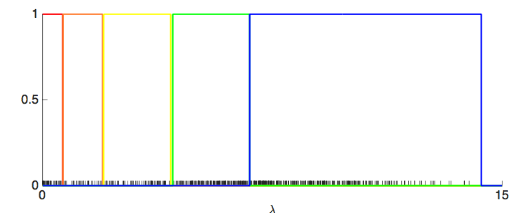


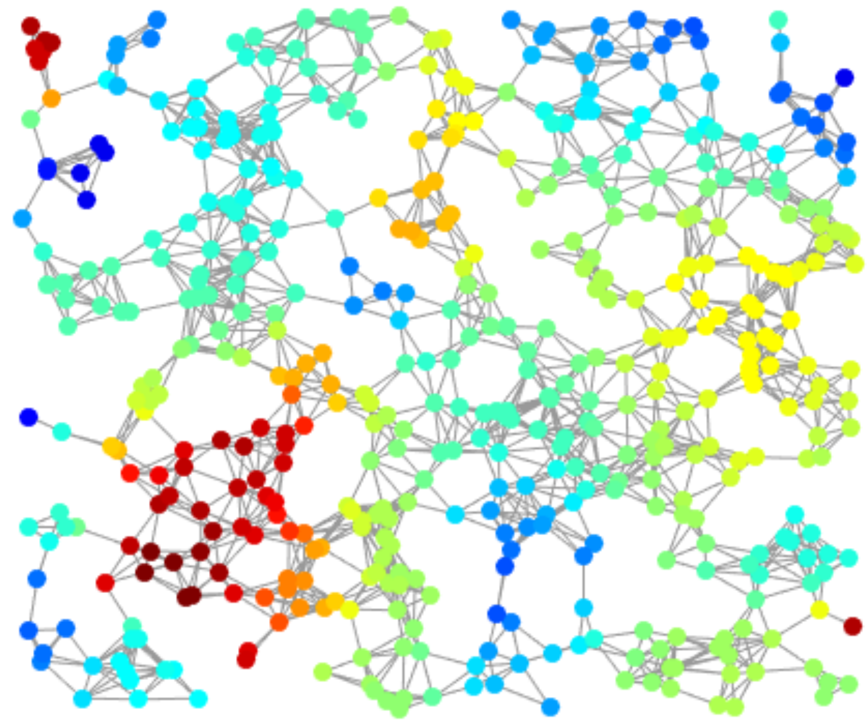
Chen et al., "Discrete signal processing on graphs: sampling theory," TSP, 2015

Sampling and Interpolation

- How to sample a graph signal and interpolate from the samples?
- How to choose the samples depends on your prior knowledge of the data
- Subset V_s of vertices is a <u>uniqueness set</u> for a subspace P iff:

  If two signals in the subspace P have the same values on the vertices in the uniqueness set, then they are the same signal

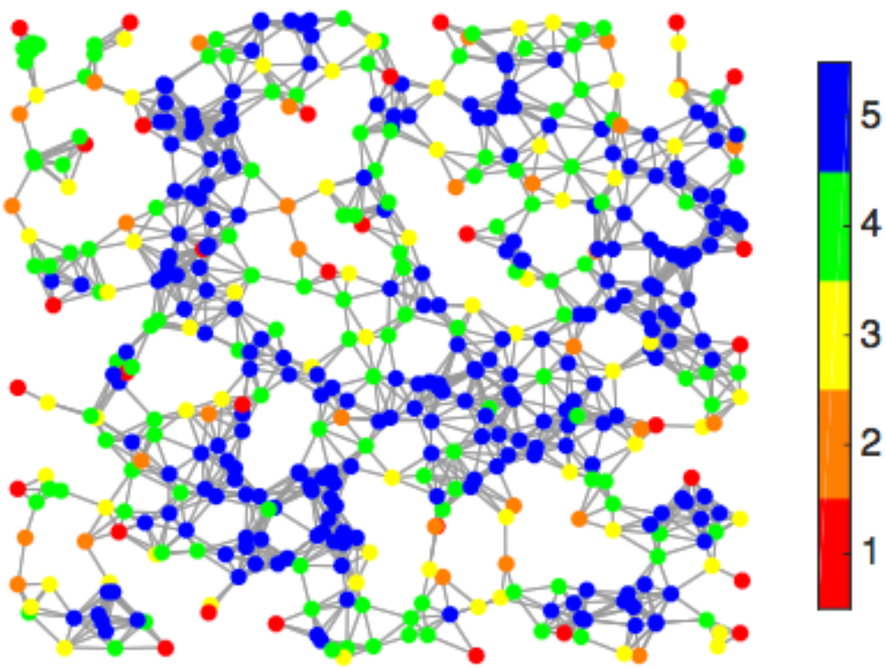


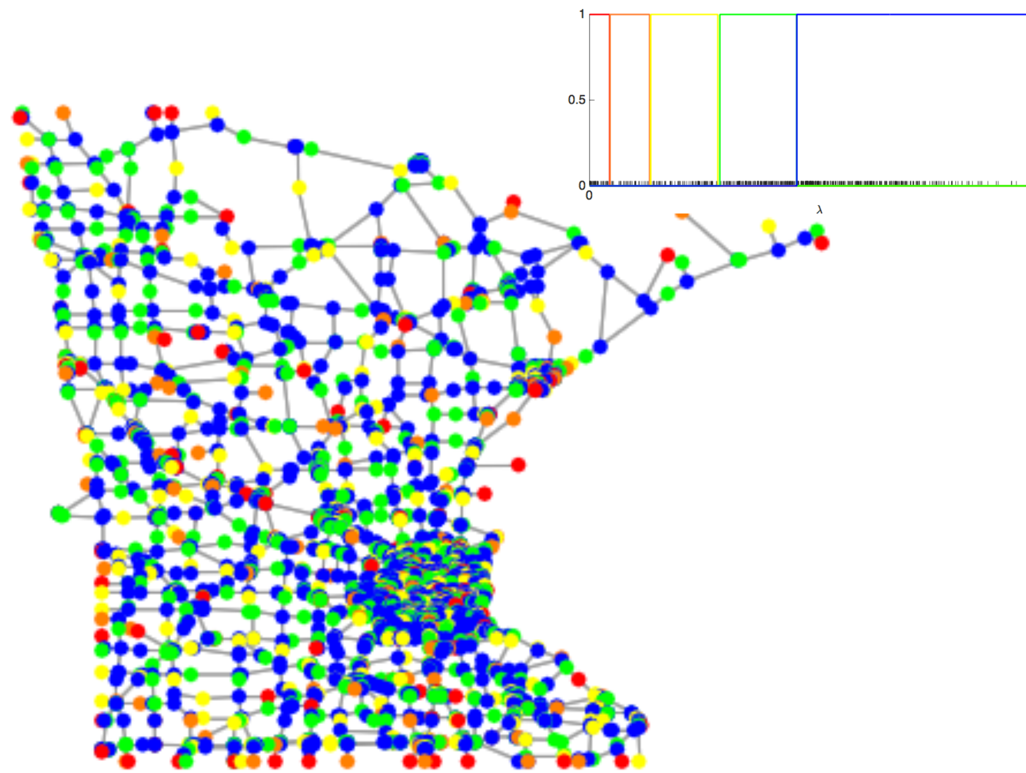


Can we recover all 500 values of this signal from 30 measurements? If so, where should we take those measurements?

Pesenson, "Sampling in Paley-Wiener spaces on combinatorial graphs," Trans. AMS, 2008

Partition into M Uniqueness Sets for Ideal Filter Bank Subspaces



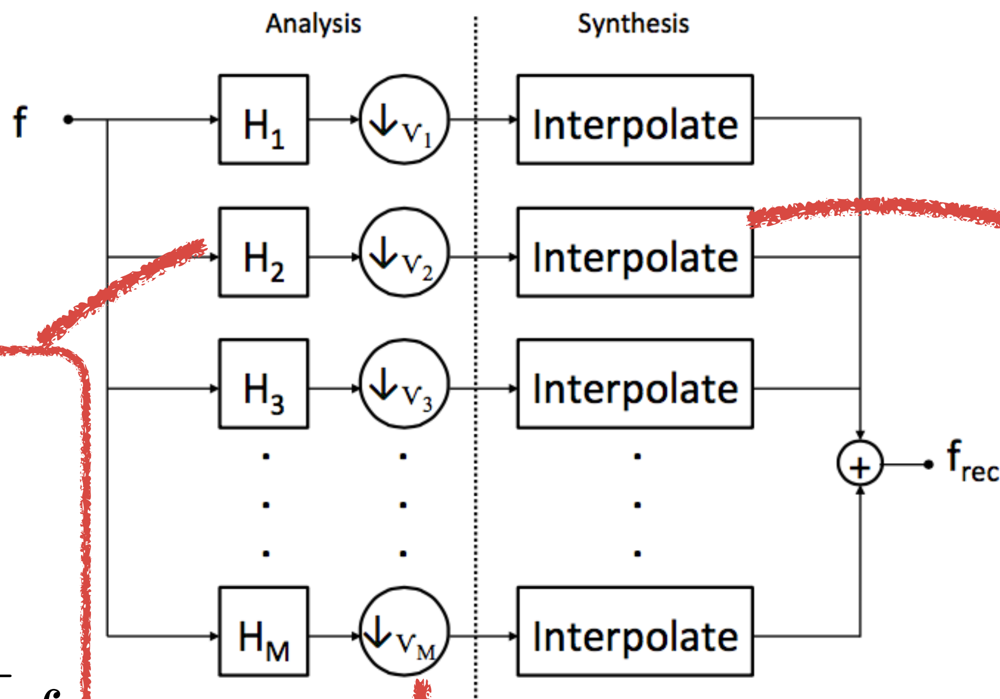


- Initialize selection for each band via greedy algorithms
- Refine to ensure the partition with techniques initially discovered in the context of matroid theory

Greene and Magnanti, "Some abstract pivot algorithms," *SIAM J. Appl. Math.*, 1975

Fast M-CSFB

Improving the Computational Efficiency for Large, Sparse Graphs



Polynomial approximation:

$$H_m f$$

$$= U h_m(\Lambda) U^\top f$$

$$\approx U \tilde{h}_m(\Lambda) U^\top f$$

$$= \tilde{h}_m(\mathcal{L}) f$$

Replace with non-uniform random sampling

Recover graph Fourier coefficients:

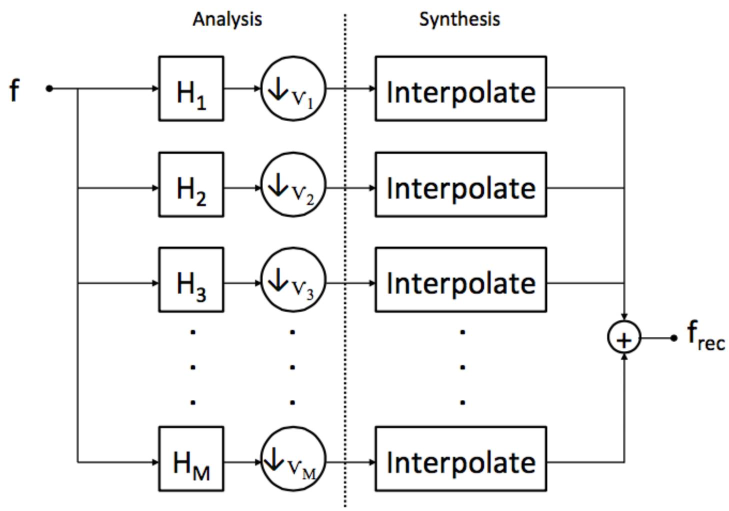
$$U_{\mathcal{V}_m, \mathcal{R}_m} x = y_{\mathcal{V}_m}$$

Interpolate:

$$\tilde{f}_m = U_{:, \mathcal{R}_m} x$$

Replace with convex optimization

Improving the Computational Efficiency for Large, Sparse Graphs

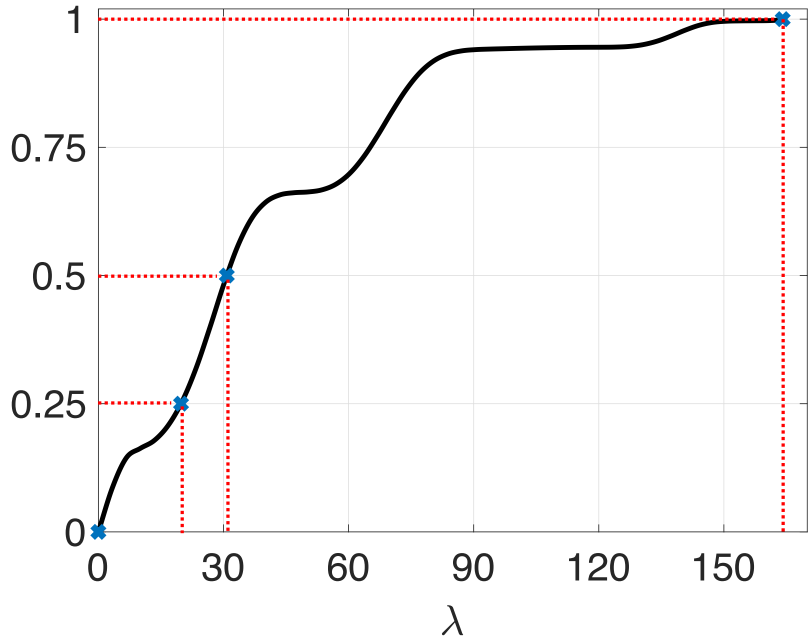


- How to design the filters to be more amenable to polynomial approximation?
- How to allocate the N samples across the channels?
- How to choose the non-uniform random sampling distribution for each downsampling set?
- How to regularize the interpolation?

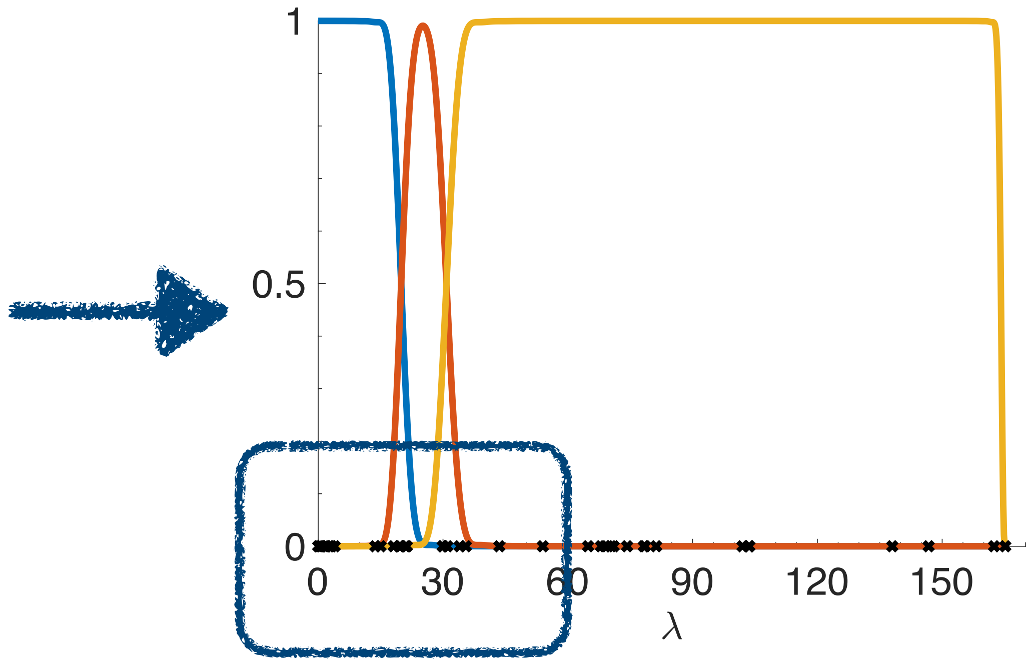
Can we improve the reconstruction error due to numerical approximations if we adapt our answers to these questions to the signal f?

Filter Bank Design

Estimate of Spectral CDF



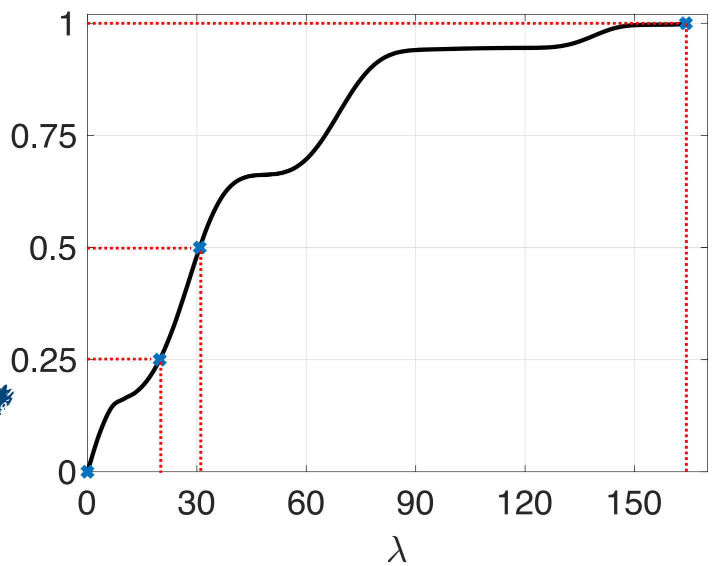
Logarithmically Spaced, Spectrum-Adapted Filter Bank



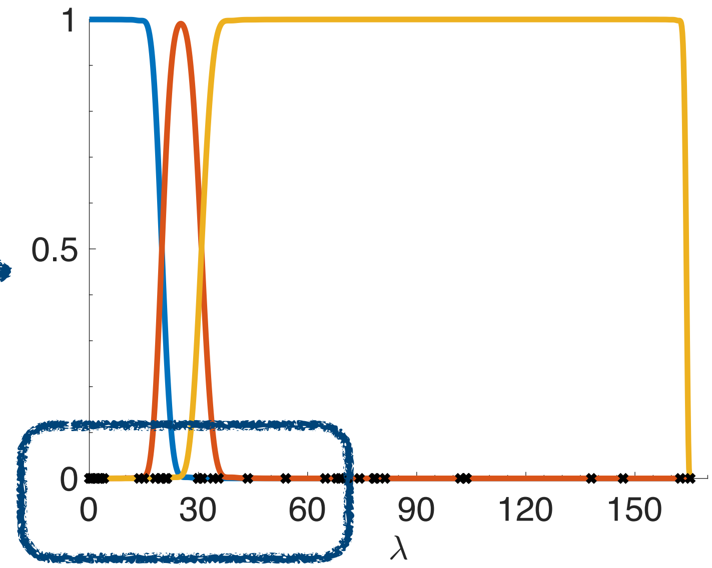
Lin, Saad, and Yang, "Approximating spectral densities of large matrices," SIAM Review, 2016

Filter Bank Design

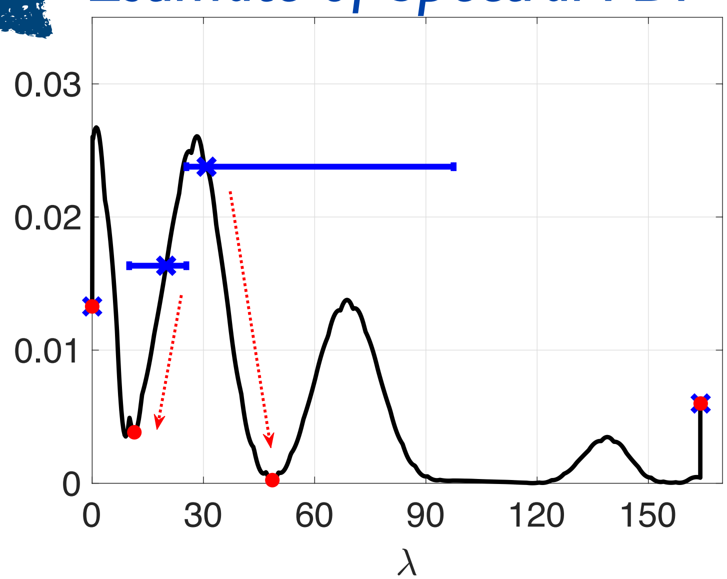
Estimate of Spectral CDF



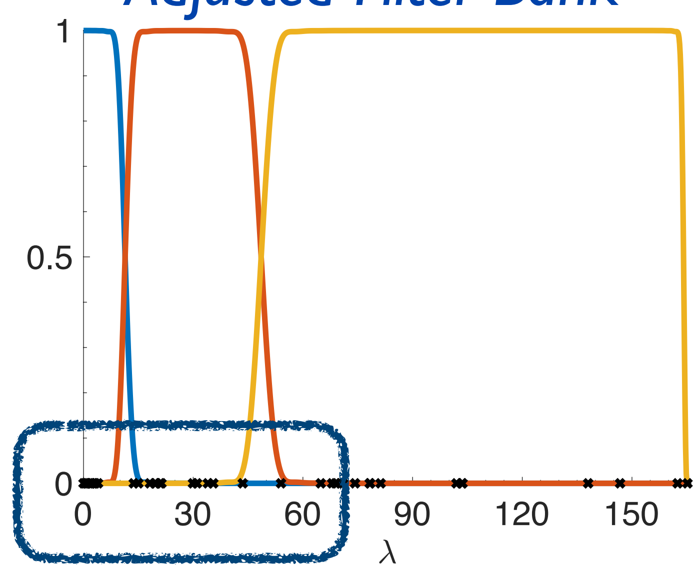
Initial Filter Bank



Estimate of Spectral PDF

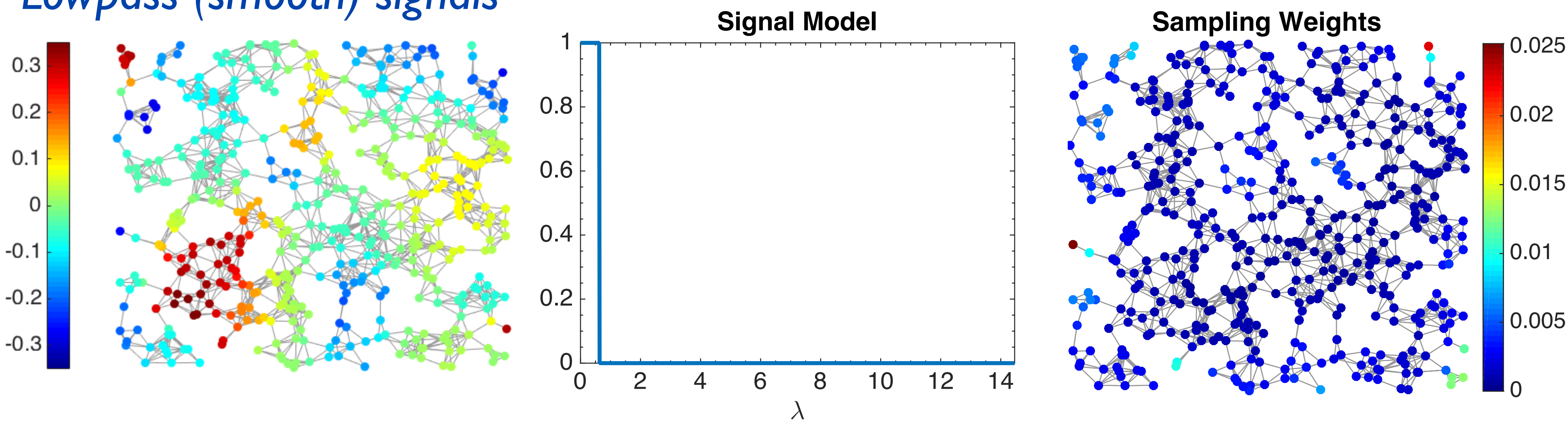


Adjusted Filter Bank

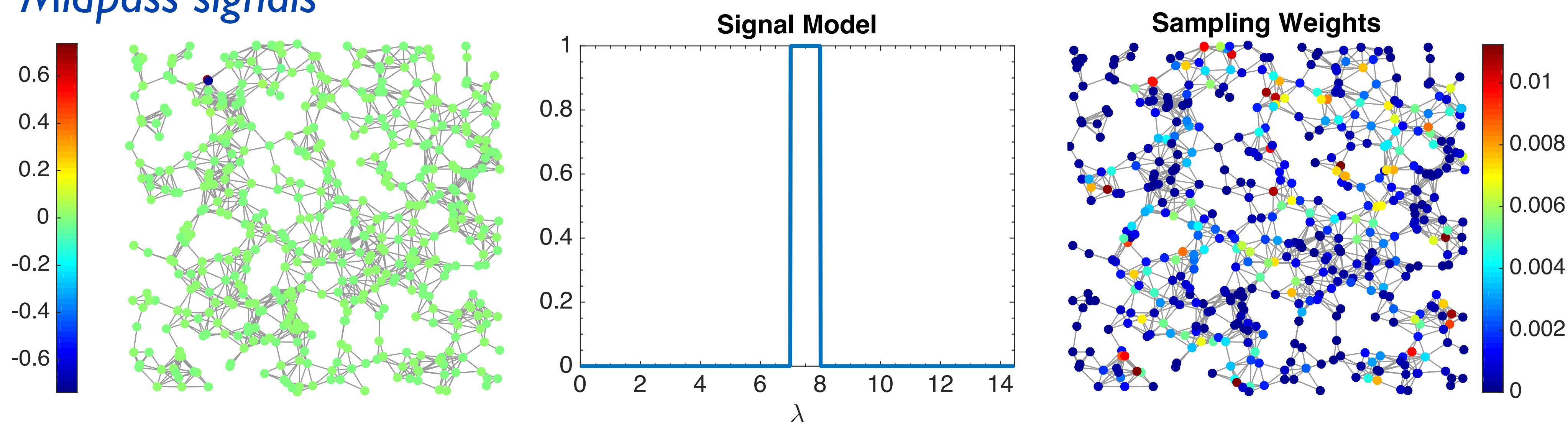


Non-Uniform Random Sampling

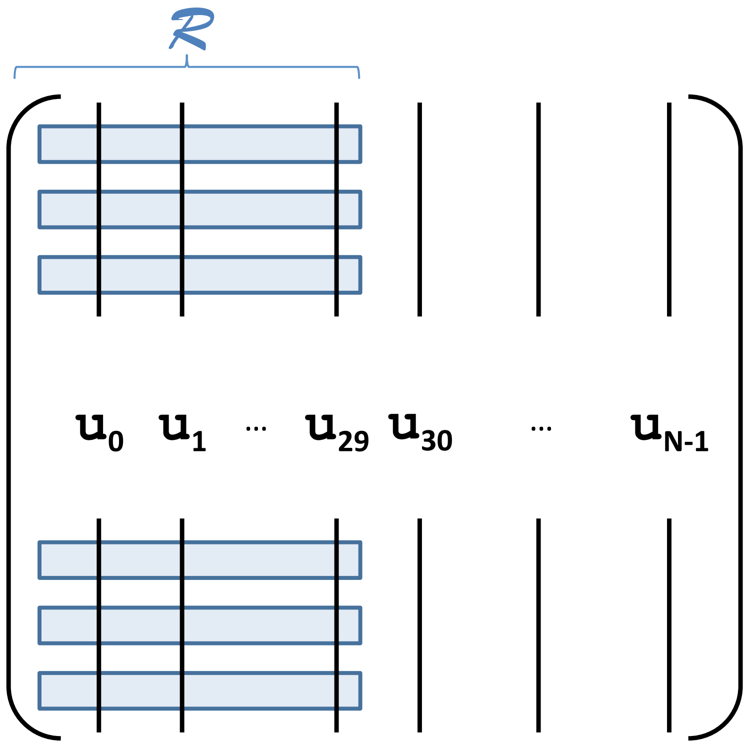
Lowpass (smooth) signals

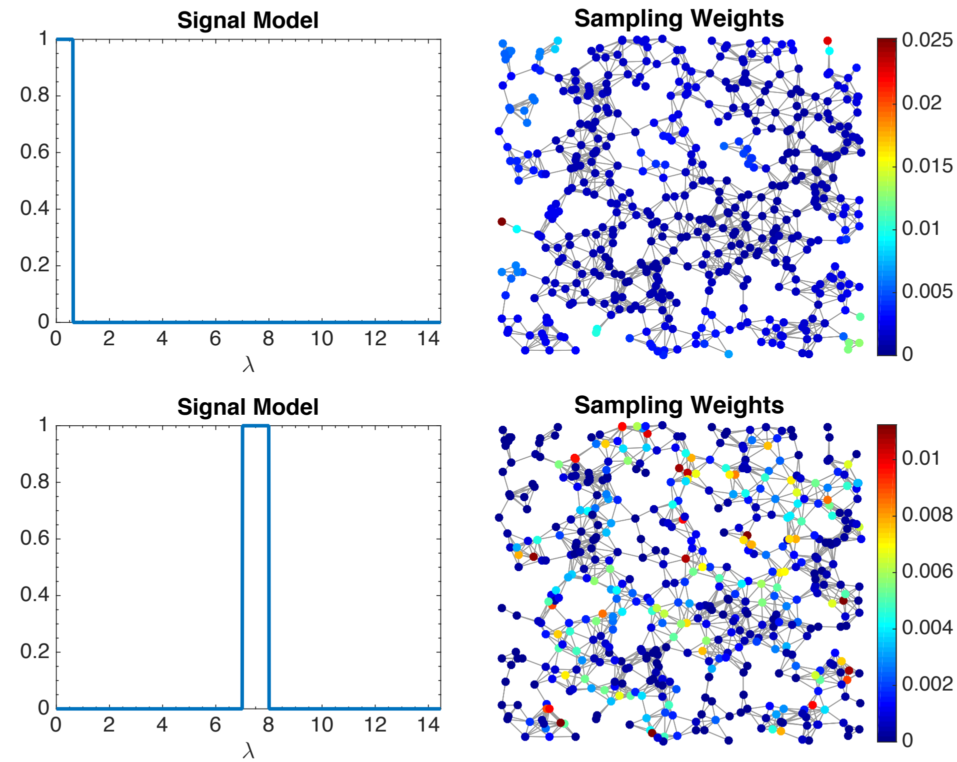


Midpass signals



Non-Uniform Random Sampling





$$p_i \sim ||U_{:,\mathcal{R}}^\top \delta_i||_2^2$$

$$\approx \frac{1}{J}\sum_{j=1}^{J}[\tilde{h}(\mathcal{L})r^{(j)}(i)]^2$$

approximation of the filter describing the signal model

independent random entries that follow a standard normal dist.

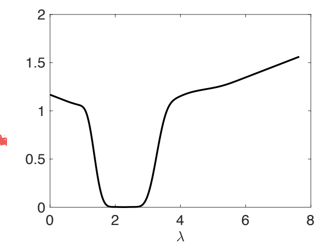
Puy et al., "Random sampling of band limited signals on graphs," ACHA, 2016

Efficient Interpolation

downsampling operator

approximate by convex optimization problem

$$\min_{z \in \text{col}(U_{:,\mathcal{R}_m})} \quad ||\Omega_{m,\mathcal{V}_m}^{-1/2}(M_m z - y_{\mathcal{V}_m})||_2^2$$

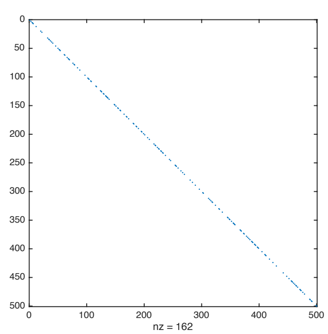


signal model space

$$\min_{z \in \mathbb{R}^N} \left\{ \gamma ||\Omega_{m,\mathcal{V}_m}^{-1/2}(M_m z - y_{\mathcal{V}_m})||_2^2 + z^\top \varphi_m(\mathcal{L}) z \right\}$$

optimality condition

$$\left(\gamma M_m^\top \Omega_{m,\mathcal{V}_m}^{-1} M_m + \varphi_m(\mathcal{L})\right) z = \gamma M_m^\top \Omega_{m,\mathcal{V}_m}^{-1} y_{\mathcal{V}_m}$$



solve with preconditioned conjugate gradient

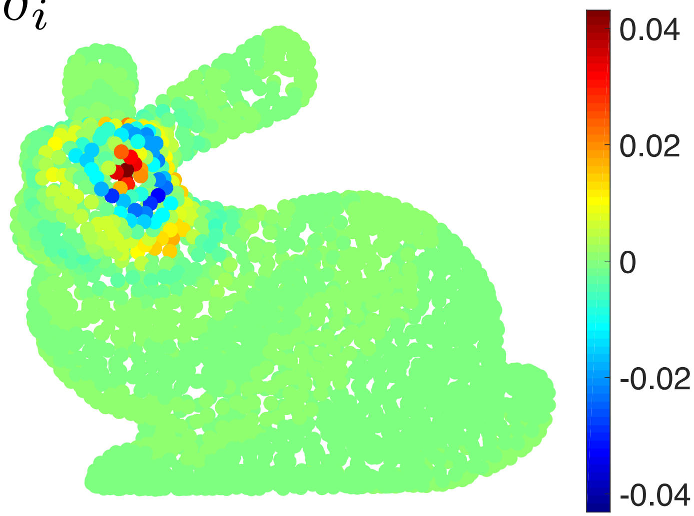
preconditioner:

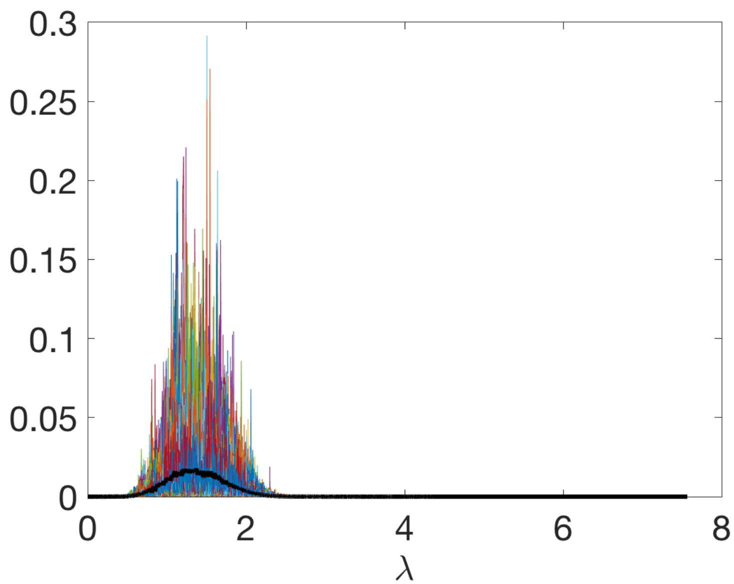
$$\text{diag}\left(1 + \frac{\gamma}{p_i} \mathbb{1}_{\{i \in \mathcal{V}_m\}}\right)$$

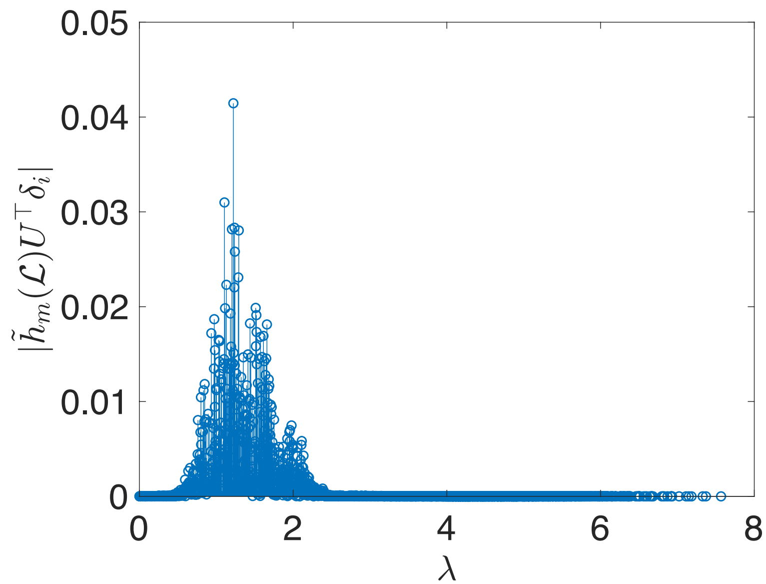
Puy et al., "Random sampling of band limited signals on graphs," ACHA, 2016

Joint Localization of Atoms

- Dictionary atoms are of the form $\tilde{h}_m(\mathcal{L})\delta_i$
- Localized within K hops of center vertex
- As K increases, become more concentrated in spectral domain

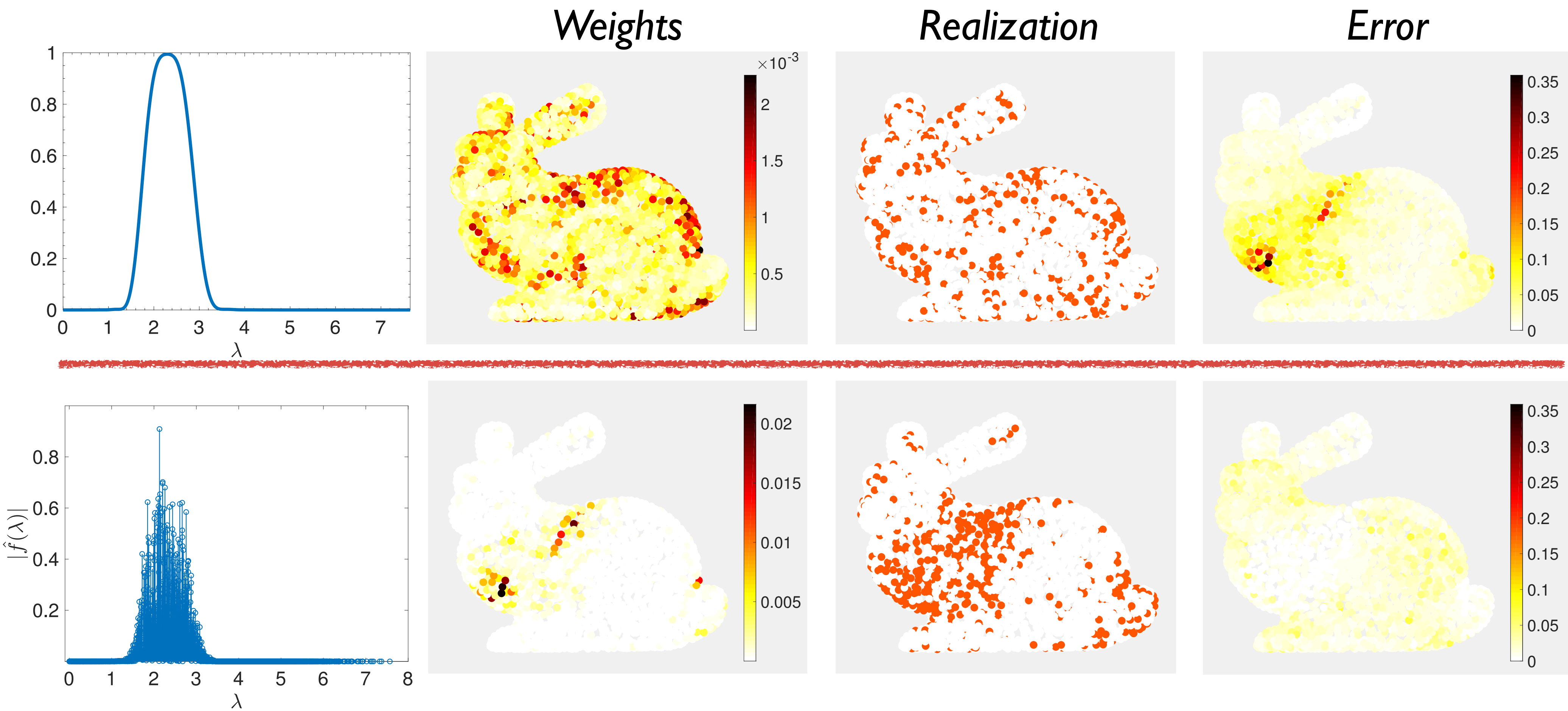


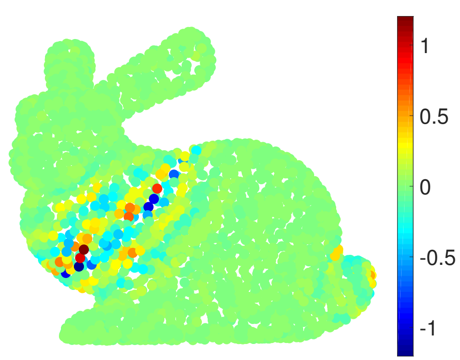




Fast, Signal-Adapted M-CSFB

Adapting the Sampling Weights to the Signal

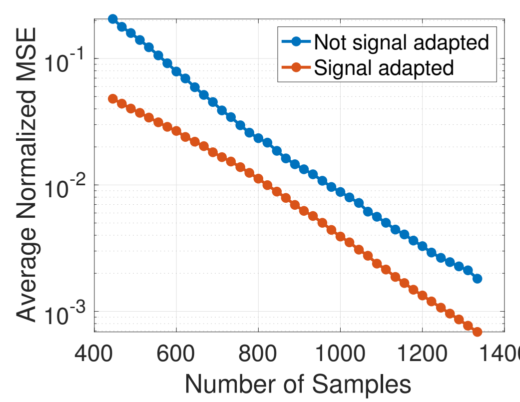




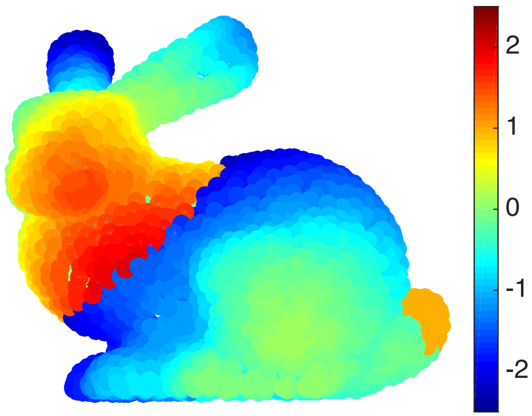
Multiply weights by

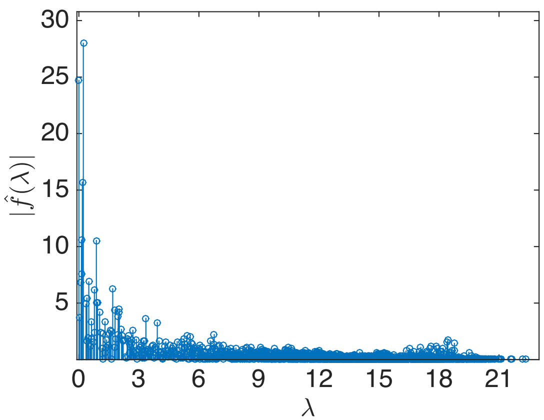
$$\log(1 + |(\tilde{h}_m(\mathcal{L})f)(i)|)$$

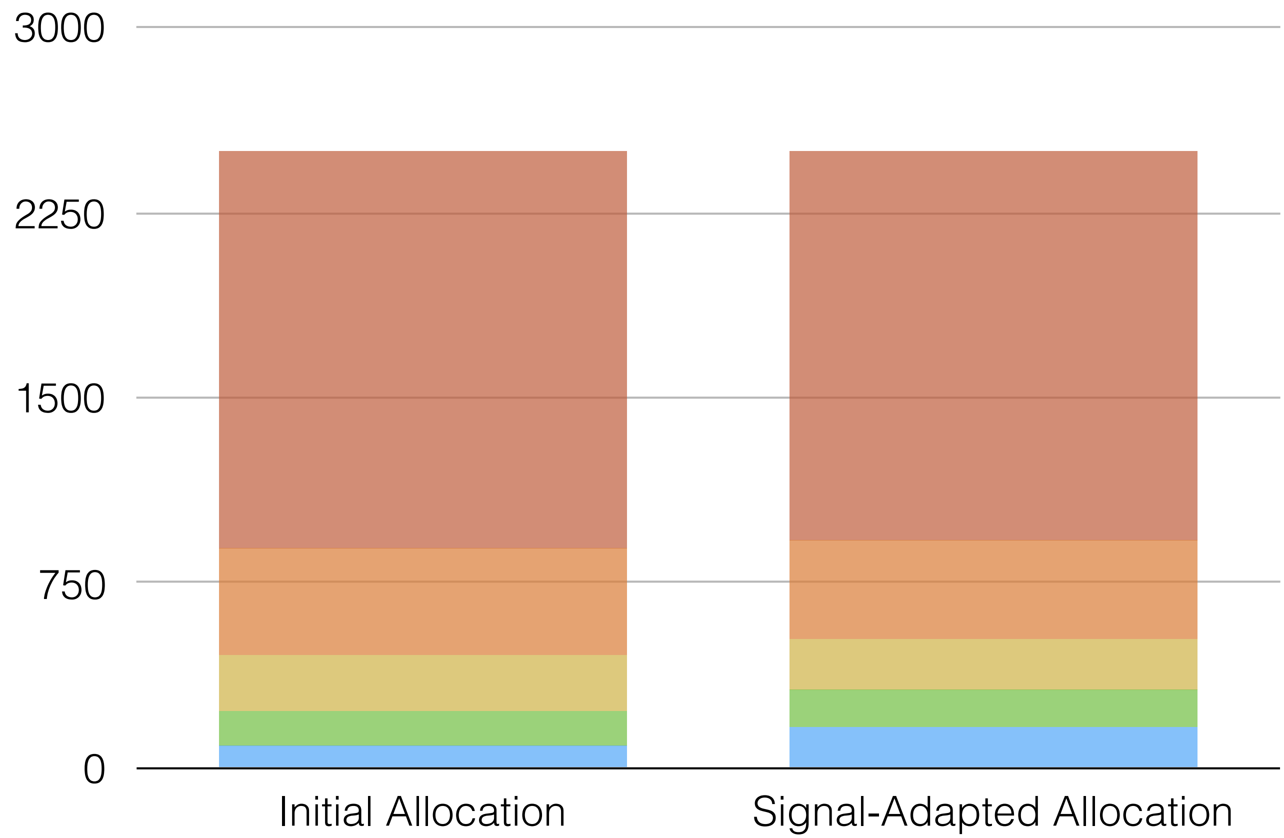
and renormalize



Adapting the Allocation of Samples



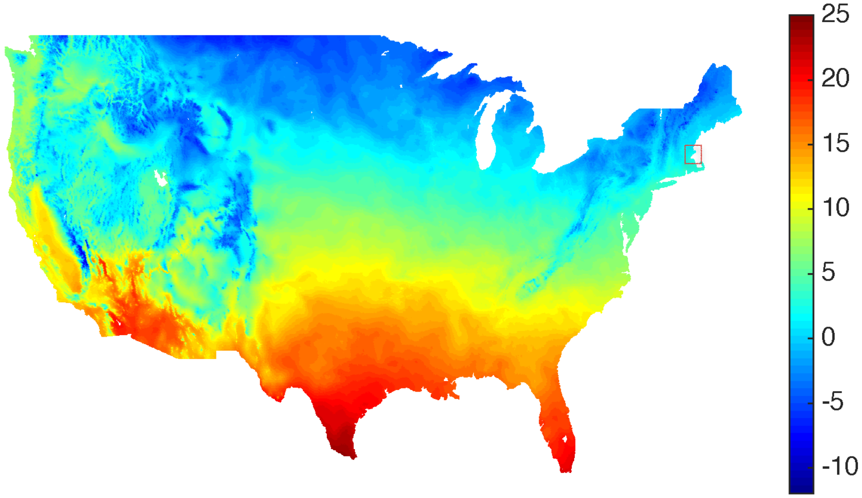


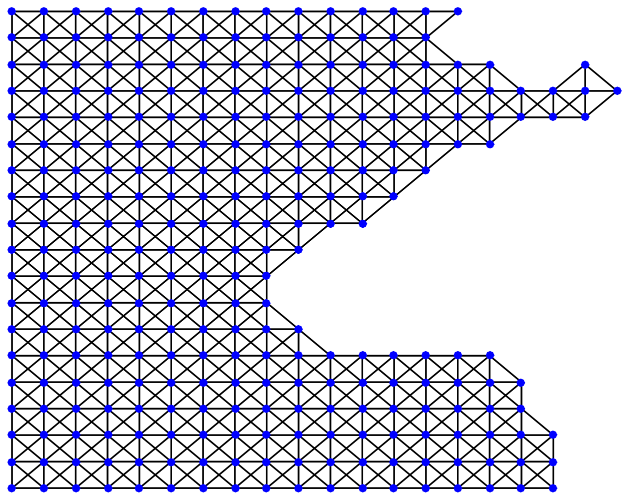


- Initial allocation proportional to $\mathrm{Trace}(X^\top \tilde{h}_m(\mathcal{L})X)$
- Multiply by $\log(1 + ||\tilde{h}_m(\mathcal{L})f||)$ and renormalize

Computation Times and Reconstruction Errors

	Sensor Network $N = 500$ $\|\mathcal{E}\| = 2,050$			Bunny $N = 2,503$ $\|\mathcal{E}\| = 13,726$			Andrianov net25 Graph $N = 9,520$ $\|\mathcal{E}\| = 195,841$			Community Graph $N = 25,000$ $\|\mathcal{E}\| = 480,459$			Temperatures $N = 469,404$ $\|\mathcal{E}\| = 1,865,415$		
	Anal. Time	Synth. Time	Rec. NMSE	Anal. Time	Synth. Time	Rec. NMSE	Anal. Time	Synth. Time	Rec. NMSE	Anal. Time	Synth. Time	Rec. NMSE	Anal. Time	Synth. Time	Rec. NMSE
Graph Fourier Transform	0.1	0.01	5.4e-30	9.8	0.02	2.5e-29	295.7	0.08	1.4e-28	8544.8	0.6	4.5e-28	NA	NA	NA
Exact M-CSFB	2.2	0.06	7.8e-30	380.4	0.1	7.8e-23	NA	NA	NA	NA	NA	NA	NA	NA	NA
Diffusion Wavelets [8]	8.5	0.03	1.2e-30	313.9	0.02	1.2e-29	14354	0.3	1.0e-26	NA	NA	NA	NA	NA	NA
Graph-QMF [14]	0.6	0.1	5.4e-8	4.9	3.4	3.2e-8	38.4	21.0	3.3e-9	1062.7	978.0	6.0e-8	NA	NA	NA
Fast M-CSFB (A)	0.6	0.5	6.8e-2	0.8	0.9	8.2e-2	2.3	3.1	1.6e-1	2.8	12.4	2.2e-1	55.1	94.5	1.4e-2
Fast M-CSFB (B)	0.7	1.0	9.2e-2	0.9	3.7	3.3e-2	1.4	12.1	1.4e-1	4.4	71.7	1.5e-1	91.6	874.3	7.0e-3
Signal-Adapted Fast M-CSFB (A)	0.7	0.5	3.8e-2	0.8	0.9	3.4e-2	0.8	2.2	6.7e-2	2.8	9.9	1.2e-1	47.6	98.4	1.7e-3
Signal-Adapted Fast M-CSFB (B)	0.7	1.1	2.4e-2	0.9	3.6	1.2e-2	1.3	9.7	7.7e-2	4.4	71.1	7.9e-2	81.2	976.0	6.6e-4





Discussion

- Computational complexity of setup and analysis is $\mathcal{O}(JK|\mathcal{E}|)$
  - Leverage single computation of $\{\bar{T}_k(\mathcal{L})X\}_{k=0,1,\ldots,K}$ to estimate (i) spectral density for filter bank design, (ii) number of samples for each band, and (iii) non-uniform sampling distributions
- Can be viewed as a fast graph Fourier transform with coarser resolution in the spectral domain
- On the other hand, atoms of the proposed transform can be viewed as a subset of the atoms of a spectral graph wavelet transform (with different filters)