A Windowed Graph Fourier Transform

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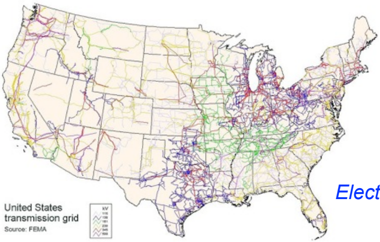
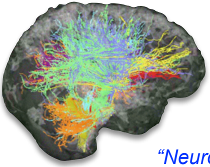
Signal Processing on Graphs

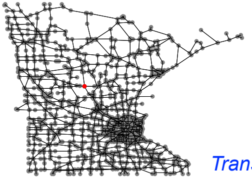


Electrical Network

Social Network

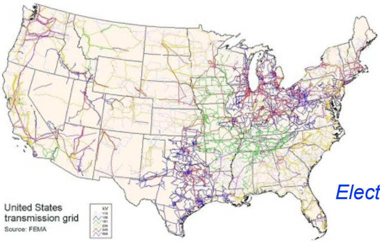


"Neuronal" Network



Transportation Network

Signal Processing on Graphs



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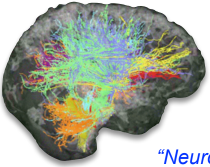

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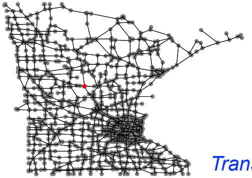

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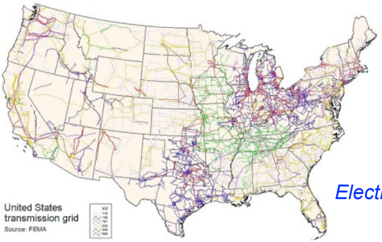
Wavelets on Graphs

- Diffusion wavelets (Coifman and Maggioni, 2006)
- Spectral graph wavelets (Hammond *et al.*, 2011)
- Wavelet filter banks (Narang and Ortega, 2012)

Signal Processing on Graphs



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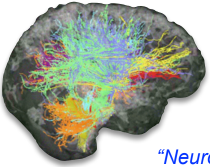

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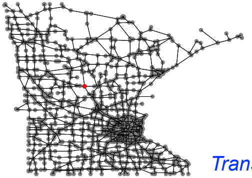

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Our approach here: extend some classical time-frequency techniques to the graph setting

Classical Time-Frequency Analysis

- Localized Fourier analysis – joint descriptions of signals' temporal and spectral behavior
- Time-frequency transforms reveal underlying structure in signal, enabling efficient information extraction, regularization in ill-posed inverse problems, etc.
  - Localized oscillations appear frequently in audio processing, vibration analysis, radar detection, etc.

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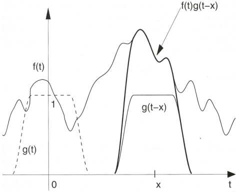
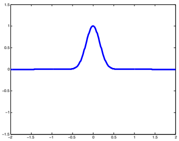


Source: Gröchenig, 2001

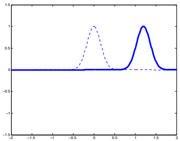
- Windowed Fourier transform of $f \in L^2(\mathbb{R})$:

$$Sf(u,\xi) := \langle f, g_{u,\xi} \rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t-u)}e^{-2\pi i \xi t}dt$$

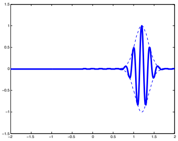
- The atoms $g_{u,\xi}$ are localized in time and frequency:



Translation T_u $\Longrightarrow$



Modulation M_ξ $\Longrightarrow$



The Essence of the Problem

Question: Why can't we just apply classical time-frequency and time-scale techniques to signals on graphs?

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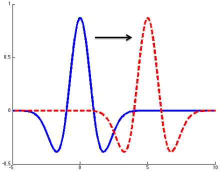
- Weighted graphs are irregular structures that lack a shift-invariant notion of translation:



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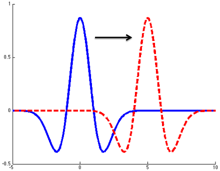

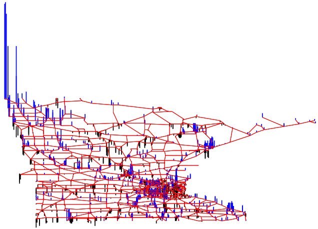

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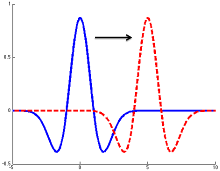
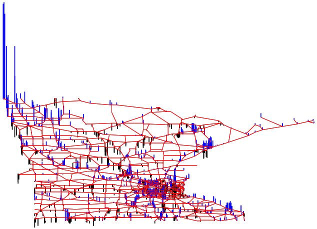




- Our objectives:
  - Develop generalized notions of convolution, translation, and modulation in the graph setting
  - Leverage these to define vertex-frequency transforms that enable us to efficiently extract information from high-dimensional data on graphs

Outline

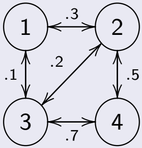
Spectral Graph Theory Notation

- Connected, undirected, weighted graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, W\}$
- Degree matrix D: zeros except diagonals, which are sums of weights of edges incident to corresponding node
- Non-normalized Laplacian: $\mathcal{L} := D - W$
- Complete set of orthonormal eigenvectors and associated real, non-negative eigenvalues:

$$\mathcal{L}\chi_\ell = \lambda_\ell \chi_\ell,$$

ordered w.l.o.g. s.t.

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 ... \leq \lambda_{N-1} := \lambda_{\max}$$

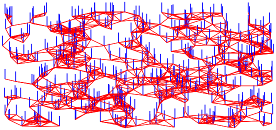


$$W = \begin{bmatrix} 0 & .3 & .1 & 0 \\ .3 & 0 & .2 & .5 \\ .1 & .2 & 0 & .7 \\ 0 & .5 & .7 & 0 \end{bmatrix}$$

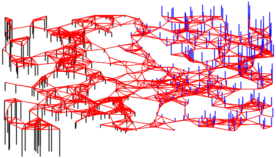
$$D = \begin{bmatrix} .4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1.2 \end{bmatrix}$$

Graph Laplacian Eigenvectors

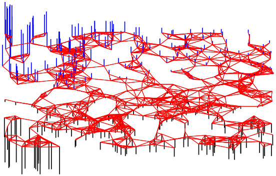
- Values of eigenvectors associated with lower frequencies (low λ_ℓ) change less rapidly across connected vertices



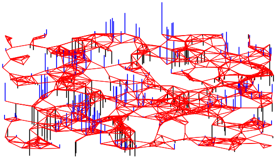
χ_0



χ_1



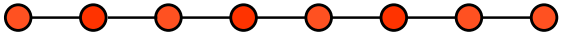
χ_2



χ_{50}

Graph Laplacian Eigenvectors

Special Case – Path Graph



Graph Laplacian Eigenvectors
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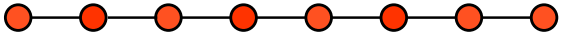
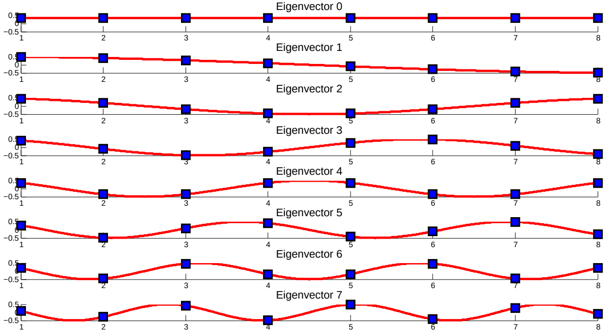


- $\lambda_\ell = 2 - 2\cos\left(\frac{\pi\ell}{N}\right)$
- $\chi_0(i) = \frac{1}{\sqrt{N}},\ \chi_\ell(i) = \sqrt{\frac{2}{N}}\cos\left(\frac{\pi\ell(i-0.5)}{N}\right),\ \ell = 1, 2, \ldots, N-1$



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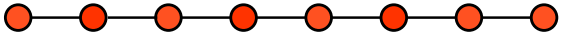

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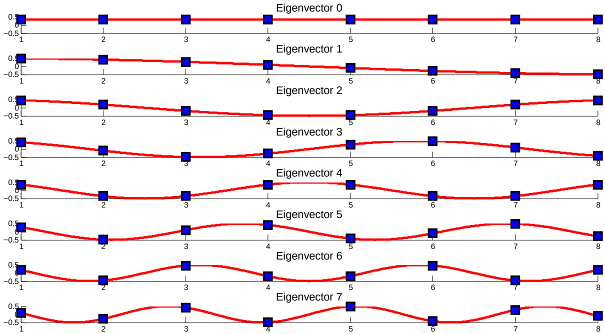
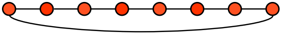


$\begin{bmatrix} | & & | \\ \chi_0 & \cdots & \chi_{N-1} \\ | & & | \end{bmatrix}$ is the Discrete Cosine Transform matrix (DCT-II, Strang, 1999), which is used in JPEG image compression

Graph Laplacian Eigenvectors
Special Case – Ring Graph



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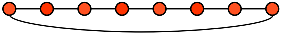

- (Unordered) Laplacian eigenvalues: $\lambda_\ell = 2 - 2\cos\left(\frac{2\ell\pi}{N}\right)$

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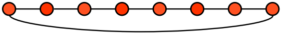

- (Unordered) Laplacian eigenvalues: $\lambda_\ell = 2 - 2\cos\left(\frac{2\ell\pi}{N}\right)$
- One possible choice of orthogonal Laplacian eigenvectors:

$$\chi_\ell = \left[1, \omega^\ell, \omega^{2\ell}, \ldots, \omega^{(N-1)\ell}\right], \text{ where } \omega = e^{\frac{2\pi j}{N}}$$

- $\begin{bmatrix} | & & | \\ \chi_0 & \cdots & \chi_{N-1} \\ | & & | \end{bmatrix}$ is the Discrete Fourier Transform (DFT) matrix

Graph Fourier Transform

- Fourier transform: expansion of f in terms of the eigenfunctions of the Laplacian / graph Laplacian

Functions on the Real Line

FOURIER TRANSFORM

$$\hat{f}(\xi) = \langle f, e^{2\pi i \xi t} \rangle = \int_{\mathbb{R}} f(t) e^{-2\pi i \xi t} \, dt$$

INVERSE FOURIER TRANSFORM

$$f(t) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi t} \, d\xi$$

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Functions on the Vertices of a Graph

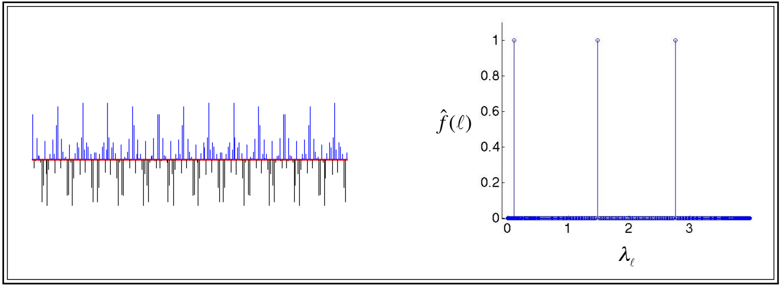
<u>GRAPH FOURIER TRANSFORM</u>

$$\hat{f}(\ell) = \langle f, \chi_\ell \rangle = \sum_{n=1}^{N} f(n) \chi_\ell^*(n)$$

<u>INVERSE GRAPH FOURIER TRANSFORM</u>

$$f(n) = \sum_{\ell=0}^{N-1} \hat{f}(\ell) \chi_\ell(n)$$

Signals on Graphs in Two Domains



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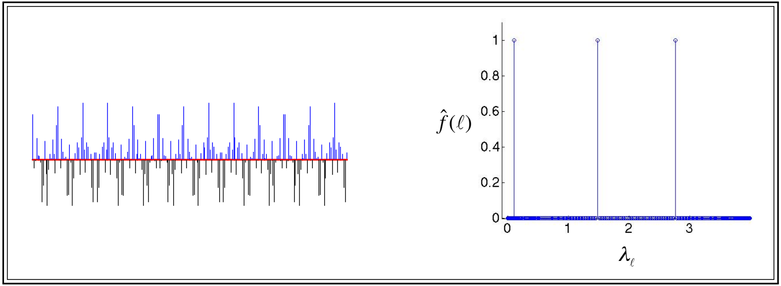

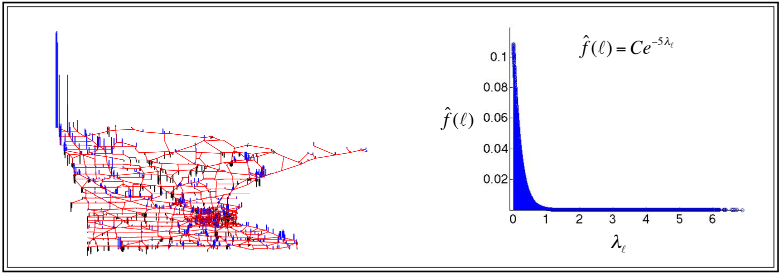

A Generalized Convolution Product for Signals on Graphs

- Convolution in the time (vertex) domain is multiplication in the Fourier (graph spectral) domain

Functions on the Real Line

For $f, g \in L^2(\mathbb{R})$,

$$(f * g)(t) := \int_{\mathbb{R}} f(\tau)g(t-\tau)d\tau,$$

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$$(f * g)(n) = \sum_{\ell=0}^{N-1} \hat{f}(\ell) \hat{g}(\ell) \chi_\ell(n)$$

- This generalized convolution product inherits properties such as commutativity, distributivity, and associativity

Generalized Translation on Graphs

- Define generalized translation via generalized convolution with a delta

Functions on the Real Line

For $f \in L^2(\mathbb{R})$, in the weak sense

$$\begin{aligned}(T_u f)(t) &:= f(t-u) \\ &= (f * \delta_u)(t) \\ &= \int_{\mathbb{R}} \hat{f}(\xi) e^{-2\pi i \xi u} e^{2\pi i \xi t} d\xi\end{aligned}$$

Functions on the Vertices of a Graph

For $f \in \mathbb{R}^N$, we define

$$\begin{aligned}(T_i f)(n) &:= \sqrt{N}(f * \delta_i)(n) \\ &= \sqrt{N} \sum_{\ell=0}^{N-1} \hat{f}(\ell) \chi_\ell^*(i) \chi_\ell(n)\end{aligned}$$

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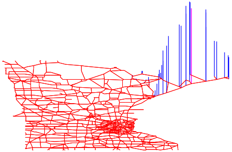
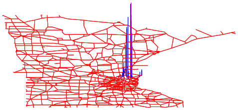
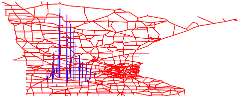


$T_{200} f$



$T_{1000} f$



$T_{2000} f$

Properties of Generalized Translation Operators on Graphs

- Some nice properties inherited from the generalized convolution:
  - $T_i T_j = T_j T_i$
  - $T_i(f * g) = (T_i f) * g = f * (T_i g)$
  - $\sum_n (T_i f)(n) = \sum_n f(n)$

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- Warning 1: Do not have the group structure of classical translation:

$$T_i T_j \neq T_{i+j}$$

- Warning 2: Unlike the classical case, generalized translation operators are not unitary:

$$\| T_i \|_2 = \max_{\ell} |\chi_\ell(i)| \,,$$

so for any $i \in \{1, 2, \ldots, N\}$,

$$1 \leq \| T_i \|_2 \leq \sqrt{N}\mu,$$

where the coherence $\mu := \max_{\ell, i} |\chi_\ell(i)|$

Generalized Modulation on Graphs

- Define generalized modulation via multiplication by a Laplacian eigenfunction / graph Laplacian eigenvector

Functions on the Real Line

For $f \in L^2(\mathbb{R})$,

$$(M_\xi f)(t) := e^{2\pi i \xi t} f(t)$$

Functions on the Vertices of a Graph

For $f \in \mathbb{R}^N$, we define

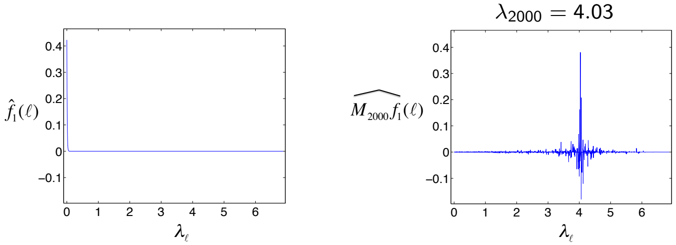
$$(M_k f)(n) := \sqrt{N} \chi_k(n) f(n)$$

- In the classical case, the modulation operator represents a translation in the Fourier domain:

$$\widehat{M_\xi f}(\omega) = \hat{f}(\omega - \xi), \ \forall \omega \in \mathbb{R}$$

Generalized Modulation as a Graph Spectral Shift?

- $\widehat{M_k \chi_0}(\lambda_\ell) = \delta_0(\lambda_\ell - \lambda_k)$, so the DC component of any signal $f \in \mathbb{R}^N$ is mapped to $\hat{f}(0)\chi_k$
- Moreover, if $\hat{f}$ is sufficiently localized around 0, then $\widehat{M_k f}$ will be localized around λ_k



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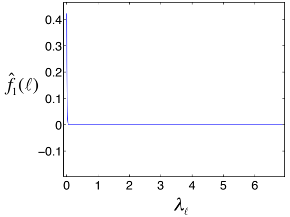
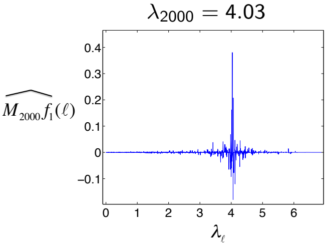
- $\widehat{M_k \chi_0}(\lambda_\ell) = \delta_0(\lambda_\ell - \lambda_k)$, so the DC component of any signal $f \in \mathbb{R}^N$ is mapped to $\hat{f}(0)\chi_k$
- Moreover, if $\hat{f}$ is sufficiently localized around 0, then $\widehat{M_k f}$ will be localized around λ_k





Theorem

If for some $\kappa > 0$, f satisfies $\frac{1}{|\hat{f}(0)|} \sum_{\ell=1}^{N-1} |\hat{f}(\ell)| \leq \frac{1}{\sqrt{N}} \left(\frac{1}{\mu + \kappa \mu^3 N} \right)$, then

$$|\widehat{M_k f}(k)| \geq \kappa |\widehat{M_k f}(\ell)| \text{ for all } \ell \neq k.$$

A Windowed Graph Fourier Transform

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Theorem (Windowed Graph Fourier Frames)

If $\hat{g}(0) \neq 0$, then $\{g_{i,k}\}_{i=1,2,\ldots,N;\ k=0,1,\ldots,N-1}$ is a frame:

$$A\|f\|_2^2 \leq \sum_{i=1}^{N} \sum_{k=0}^{N-1} |\langle f, g_{i,k} \rangle|^2 \leq B\|f\|_2^2,$$

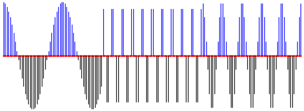
where

$$A := \min_{i \in \{1,2,\ldots,N\}} \{N\|T_i g\|_2^2\} \geq N|\hat{g}(0)|^2 > 0, \text{ and}$$

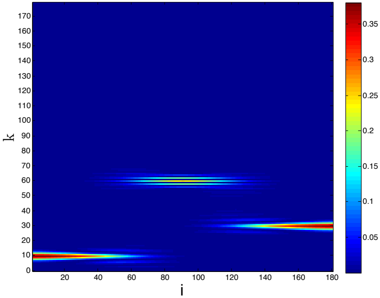
$$B := \max_{i \in \{1,2,\ldots,N\}} \{N\|T_i g\|_2^2\} \leq N^2 \mu^2 \|g\|_2^2.$$

Example 1: The Path Graph

- Signal f on the path graph comprised of three different graph Laplacian eigenvectors restricted to three different segments of the graph:

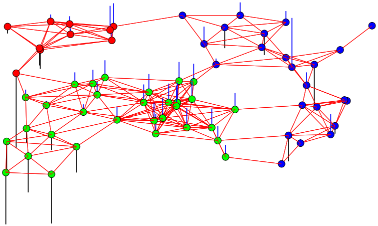


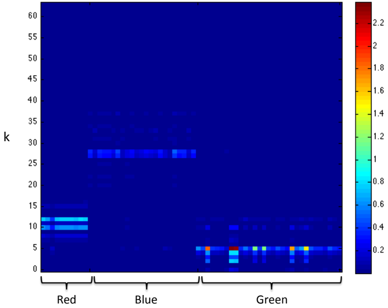
- "Spectrogram" of f showing $|Sf(i,k)|^2$, using a normalized heat kernel window with $\tau = 300$:



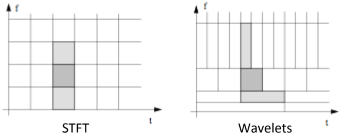
Example 2: A Random Sensor Network

- Partition a random sensor network into 3 clusters via spectral clustering
- Signal f comprised of three different graph Laplacian eigenvectors $(\chi_{10}, \chi_{27}, \chi_5)$ restricted to the three different clusters of vertices





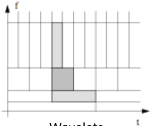
Tiling Comparison with Spectral Graph Wavelets



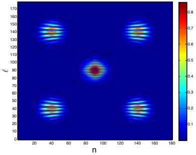
Source: Vetterli and Kovačević, 1995

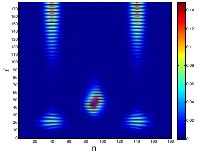
Tiling Comparison with Spectral Graph Wavelets

STFT



Wavelets





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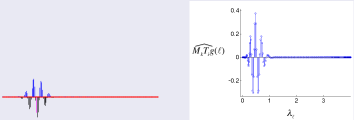
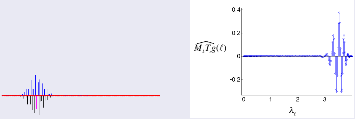
WGFT

SGWT

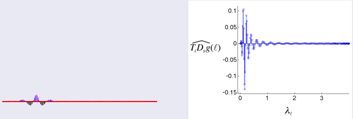
Windowed Graph Fourier Atom - Low Frequency



Windowed Graph Fourier Atom - High Frequency



Spectral Graph Wavelet - Low Frequency



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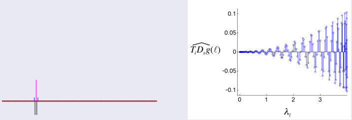
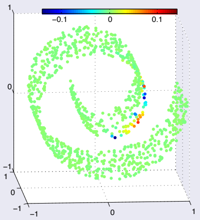
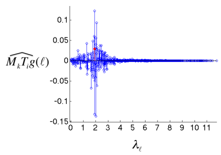


Example 3: Swiss Roll

Three different windowed graph Fourier atoms, shown in both domains:





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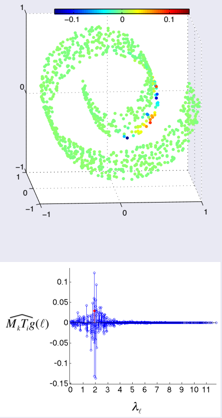

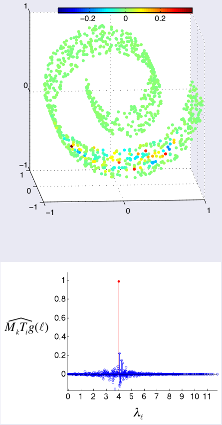

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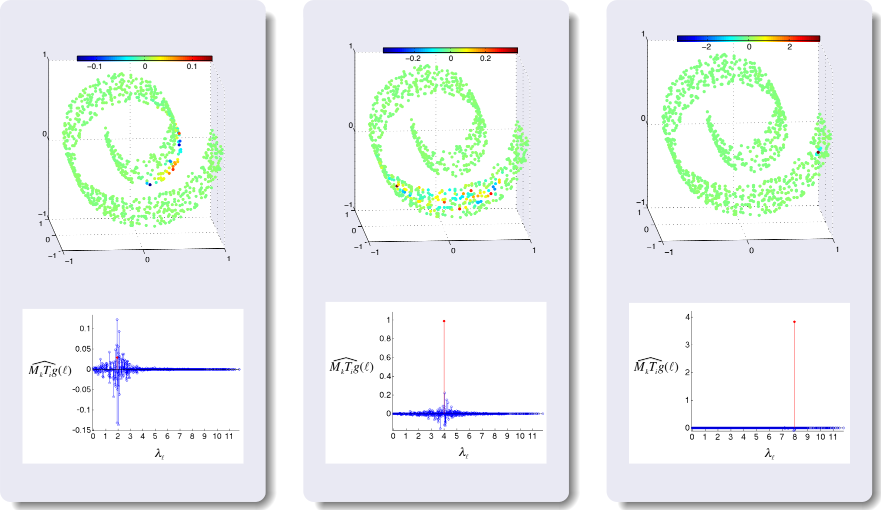

Summary and Ongoing Work

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  - Generalized translation and modulation via Laplacian eigenfunctions
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  - Just scratching the surface

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