A Windowed Graph Fourier Transform

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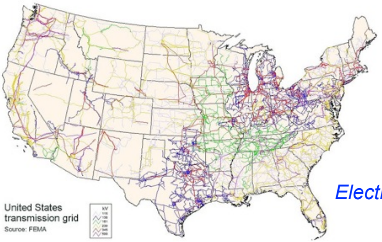
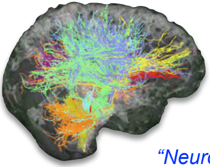
Signal Processing on Graphs

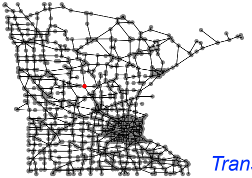


Electrical Network

Social Network



"Neuronal" Network



Transportation Network

Wavelets on Graphs

- Diffusion wavelets (Coifman and Maggioni, 2006)
- Spectral graph wavelets (Hammond *et al.*, 2011)
- Wavelet filter banks (Narang and Ortega, 2012)

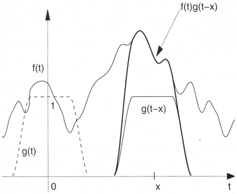
Our approach here: extend some classical time-frequency techniques to the graph setting

Classical Time-Frequency Analysis

- Localized Fourier analysis – joint descriptions of signals' temporal and spectral behavior
- Time-frequency transforms reveal underlying structure in signal, enabling efficient information extraction, regularization in ill-posed inverse problems, etc.
  - Localized oscillations appear frequently in audio processing, vibration analysis, radar detection, etc.

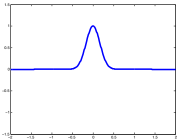
- Windowed Fourier transform of $f \in L^2(\mathbb{R})$:

$$Sf(u,\xi) := \langle f, g_{u,\xi} \rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t-u)}e^{-2\pi i \xi t}dt$$

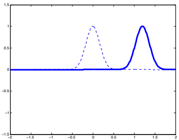


Source: Gröchenig, 2001

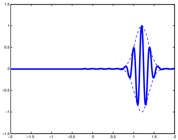
- The atoms $g_{u,\xi}$ are localized in time and frequency:



Translation T_u $\Longrightarrow$



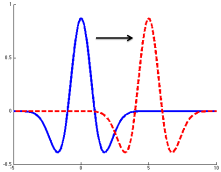
Modulation M_ξ $\Longrightarrow$

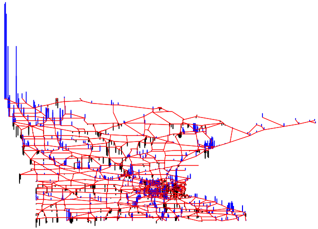


The Essence of the Problem

Question: Why can't we just apply classical time-frequency and time-scale techniques to signals on graphs?

- Weighted graphs are irregular structures that lack a shift-invariant notion of translation:





- Our objectives:
  - Develop generalized notions of convolution, translation, and modulation in the graph setting
  - Leverage these to define vertex-frequency transforms that enable us to efficiently extract information from high-dimensional data on graphs

Outline

1. Introduction
2. Spectral Graph Theory Background
3. Generalized Convolution, Translation, and Modulation
4. Windowed Graph Fourier Frames
5. Examples
6. Conclusion

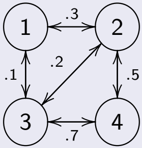
Spectral Graph Theory Notation

- Connected, undirected, weighted graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, W\}$
- Degree matrix D: zeros except diagonals, which are sums of weights of edges incident to corresponding node
- Non-normalized Laplacian: $\mathcal{L} := D - W$
- Complete set of orthonormal eigenvectors and associated real, non-negative eigenvalues:

$$\mathcal{L}\chi_\ell = \lambda_\ell \chi_\ell,$$

ordered w.l.o.g. s.t.

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 ... \leq \lambda_{N-1} := \lambda_{\max}$$

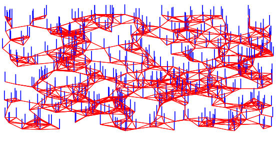


$$W = \begin{bmatrix} 0 & .3 & .1 & 0 \\ .3 & 0 & .2 & .5 \\ .1 & .2 & 0 & .7 \\ 0 & .5 & .7 & 0 \end{bmatrix}$$

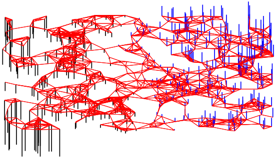
$$D = \begin{bmatrix} .4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1.2 \end{bmatrix}$$

Graph Laplacian Eigenvectors

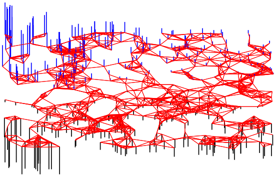
- Values of eigenvectors associated with lower frequencies (low λ_ℓ) change less rapidly across connected vertices



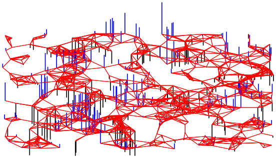
χ_0



χ_1

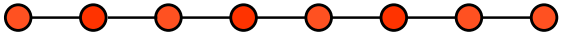


χ_2

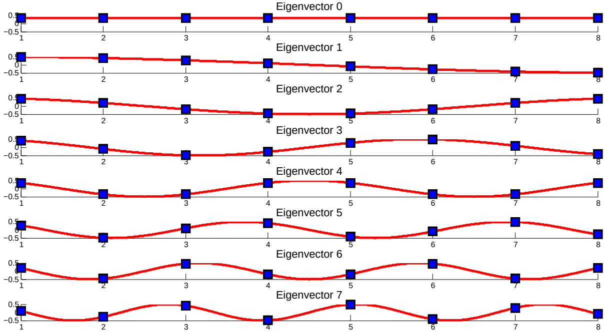


χ_{50}

Graph Laplacian Eigenvectors
Special Case – Path Graph

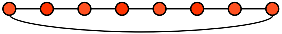


- $\lambda_\ell = 2 - 2\cos\left(\frac{\pi \ell}{N}\right)$
- $\chi_0(i) = \frac{1}{\sqrt{N}},\ \chi_\ell(i) = \sqrt{\frac{2}{N}} \cos\left(\frac{\pi \ell (i-0.5)}{N}\right),\ \ell = 1, 2, \ldots, N-1$



$$\begin{bmatrix} | & & | \\ \chi_0 & \cdots & \chi_{N-1} \\ | & & | \end{bmatrix}$$ is the Discrete Cosine Transform matrix (DCT-II, Strang, 1999), which is used in JPEG image compression

Graph Laplacian Eigenvectors
Special Case – Ring Graph



- (Unordered) Laplacian eigenvalues: $\lambda_\ell = 2 - 2\cos\left(\frac{2\ell\pi}{N}\right)$

- One possible choice of orthogonal Laplacian eigenvectors:

$$\chi_\ell = \left[1, \omega^\ell, \omega^{2\ell}, \ldots, \omega^{(N-1)\ell}\right], \text{ where } \omega = e^{\frac{2\pi j}{N}}$$

- $\begin{bmatrix} | & & | \\ \chi_0 & \cdots & \chi_{N-1} \\ | & & | \end{bmatrix}$ is the Discrete Fourier Transform (DFT) matrix

Graph Fourier Transform

- Fourier transform: expansion of f in terms of the eigenfunctions of the Laplacian / graph Laplacian

Functions on the Real Line

FOURIER TRANSFORM

$$\hat{f}(\xi) = \langle f, e^{2\pi i \xi t} \rangle = \int_{\mathbb{R}} f(t) e^{-2\pi i \xi t} \, dt$$

INVERSE FOURIER TRANSFORM

$$f(t) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi t} \, d\xi$$

Functions on the Vertices of a Graph

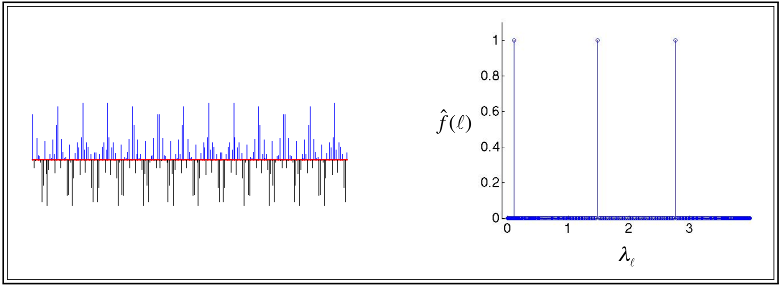
GRAPH FOURIER TRANSFORM

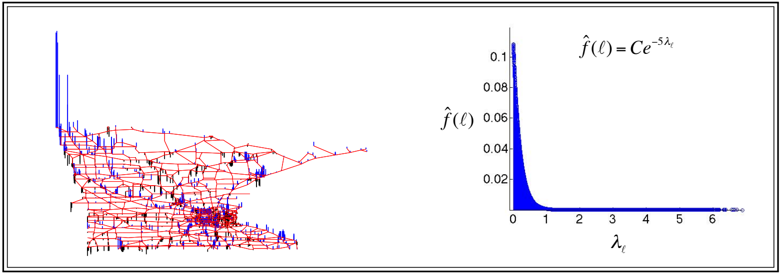
$$\hat{f}(\ell) = \langle f, \chi_\ell \rangle = \sum_{n=1}^{N} f(n) \chi_\ell^*(n)$$

INVERSE GRAPH FOURIER TRANSFORM

$$f(n) = \sum_{\ell=0}^{N-1} \hat{f}(\ell) \chi_\ell(n)$$

Signals on Graphs in Two Domains





A Generalized Convolution Product for Signals on Graphs

- Convolution in the time (vertex) domain is multiplication in the Fourier (graph spectral) domain

Functions on the Real Line

For $f, g \in L^2(\mathbb{R})$,

$$(f * g)(t) := \int_{\mathbb{R}} f(\tau) g(t - \tau) d\tau,$$

which implies

$$(f * g)(t) = \int_{\mathbb{R}} \hat{f}(\xi) \hat{g}(\xi) e^{2\pi i \xi t} d\xi$$

Functions on the Vertices of a Graph

For $f, g \in \mathbb{R}^N$, we define

$$(f * g)(n) = \sum_{\ell=0}^{N-1} \hat{f}(\ell) \hat{g}(\ell) \chi_\ell(n)$$

- This generalized convolution product inherits properties such as commutativity, distributivity, and associativity

Generalized Translation on Graphs

- Define generalized translation via generalized convolution with a delta

Functions on the Real Line

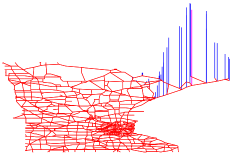
For $f \in L^2(\mathbb{R})$, in the weak sense

$$(T_u f)(t) := f(t-u) = (f * \delta_u)(t) = \int_{\mathbb{R}} \hat{f}(\xi) e^{-2\pi i \xi u} e^{2\pi i \xi t} d\xi$$

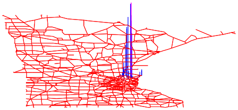
Functions on the Vertices of a Graph

For $f \in \mathbb{R}^N$, we define

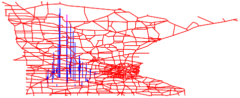
$$(T_i f)(n) := \sqrt{N}(f * \delta_i)(n) = \sqrt{N} \sum_{\ell=0}^{N-1} \hat{f}(\ell) \chi_\ell^*(i) \chi_\ell(n)$$



$T_{200} f$



$T_{1000} f$



$T_{2000} f$

Properties of Generalized Translation Operators on Graphs

- Some nice properties inherited from the generalized convolution:
  - $T_i T_j = T_j T_i$
  - $T_i(f * g) = (T_i f) * g = f * (T_i g)$
  - $\sum_n (T_i f)(n) = \sum_n f(n)$

- Warning 1: Do not have the group structure of classical translation:

$$T_i T_j \neq T_{i+j}$$

- Warning 2: Unlike the classical case, generalized translation operators are not unitary:

$$\|T_i\|_2 = \max_{\ell} |\chi_\ell(i)|,$$

so for any $i \in \{1, 2, \ldots, N\}$,

$$1 \leq \|T_i\|_2 \leq \sqrt{N}\mu,$$

where the coherence $\mu := \max_{\ell,i} |\chi_\ell(i)|$

Generalized Modulation on Graphs

- Define generalized modulation via multiplication by a Laplacian eigenfunction / graph Laplacian eigenvector

Functions on the Real Line

For $f \in L^2(\mathbb{R})$,

$$(M_\xi f)(t) := e^{2\pi i \xi t} f(t)$$

Functions on the Vertices of a Graph

For $f \in \mathbb{R}^N$, we define

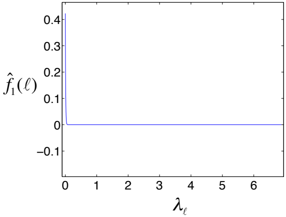
$$(M_k f)(n) := \sqrt{N} \chi_k(n) f(n)$$

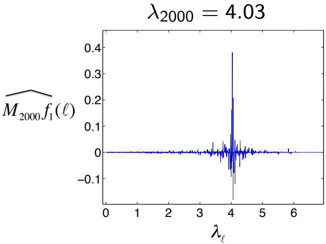
- In the classical case, the modulation operator represents a translation in the Fourier domain:

$$\widehat{M_\xi f}(\omega) = \hat{f}(\omega - \xi), \ \forall \omega \in \mathbb{R}$$

Generalized Modulation as a Graph Spectral Shift?

- $\widehat{M_k \chi_0}(\lambda_\ell) = \delta_0(\lambda_\ell - \lambda_k)$, so the DC component of any signal $f \in \mathbb{R}^N$ is mapped to $\hat{f}(0)\chi_k$
- Moreover, if $\hat{f}$ is sufficiently localized around 0, then $\widehat{M_k f}$ will be localized around λ_k





Theorem

If for some $\kappa > 0$, f satisfies $\frac{1}{|\hat{f}(0)|} \sum_{\ell=1}^{N-1} |\hat{f}(\ell)| \leq \frac{1}{\sqrt{N}} \left(\frac{1}{\mu + \kappa \mu^3 N} \right)$, then

$$|\widehat{M_k f}(k)| \geq \kappa |\widehat{M_k f}(\ell)| \text{ for all } \ell \neq k.$$

A Windowed Graph Fourier Transform

- Windowed graph Fourier atoms:

$$g_{i,k} := M_k T_i g$$

- Windowed graph Fourier transform:

$$Sf(i,k) := \langle f, g_{i,k} \rangle$$

Theorem (Windowed Graph Fourier Frames)

If $\hat{g}(0) \neq 0$, then $\{g_{i,k}\}_{i=1,2,\ldots,N;\ k=0,1,\ldots,N-1}$ is a frame:

$$A\|f\|_2^2 \leq \sum_{i=1}^{N} \sum_{k=0}^{N-1} |\langle f, g_{i,k} \rangle|^2 \leq B\|f\|_2^2,$$

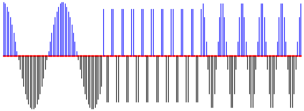
where

$$A := \min_{i \in \{1,2,\ldots,N\}} \{N\|T_i g\|_2^2\} \geq N|\hat{g}(0)|^2 > 0, \text{ and}$$

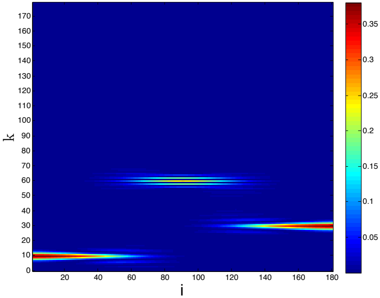
$$B := \max_{i \in \{1,2,\ldots,N\}} \{N\|T_i g\|_2^2\} \leq N^2 \mu^2 \|g\|_2^2.$$

Example 1: The Path Graph

- Signal f on the path graph comprised of three different graph Laplacian eigenvectors restricted to three different segments of the graph:

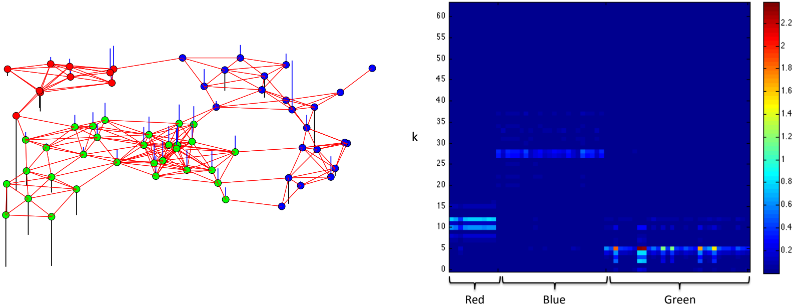


- "Spectrogram" of f showing $|Sf(i,k)|^2$, using a normalized heat kernel window with $\tau = 300$:

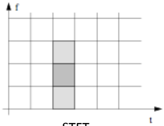


Example 2: A Random Sensor Network

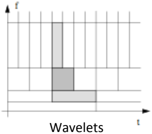
- Partition a random sensor network into 3 clusters via spectral clustering
- Signal f comprised of three different graph Laplacian eigenvectors $(\chi_{10}, \chi_{27}, \chi_5)$ restricted to the three different clusters of vertices



Tiling Comparison with Spectral Graph Wavelets

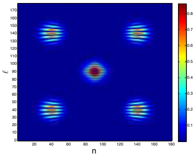


STFT

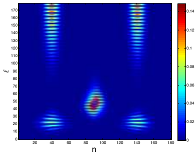


Wavelets

Source: Vetterli and Kovačević, 1995

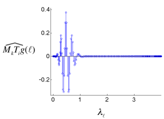


WGFT

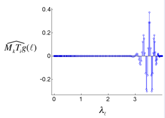


SGWT

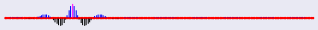
Windowed Graph Fourier Atom - Low Frequency

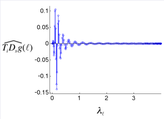


Windowed Graph Fourier Atom - High Frequency

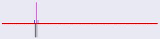


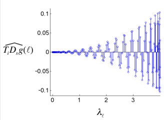
Spectral Graph Wavelet - Low Frequency





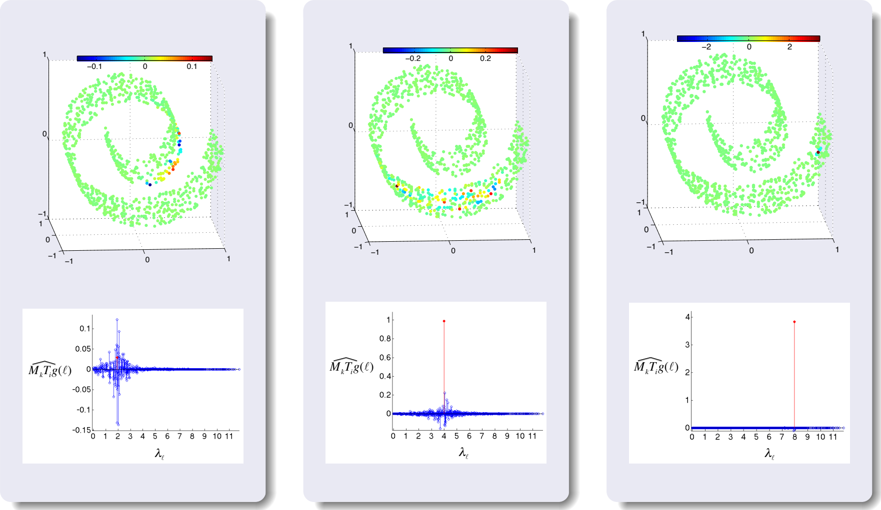
Spectral Graph Wavelet - High Frequency





Example 3: Swiss Roll

Three different windowed graph Fourier atoms, shown in both domains:



Summary and Ongoing Work

- Summary:
  - Generalized translation and modulation via Laplacian eigenfunctions
  - Leveraged these operators to design windowed graph Fourier frames
  - For the path graph or highly-structured signals, the generalized "spectrogram" matches our classical time-frequency intuition
  - Just scratching the surface
- Ongoing work:
  - Mathematical theory linking 1) structural properties of graph signals and their underlying graphs to 2) properties of the generalized operators and transform coefficients (sparsity, localization, uncertainty principles)
    - Important for optimal window design, efficient information extraction, and choosing appropriate regularization techniques for ill-posed inverse problems
  - Computationally efficient implementations