Franklin W. Olin College of Engineering
Needham, Massachusetts
Electrical and Computer Engineering

ENGR3420 – Analog and Digital Communications

Information Theory Exam

*Posted Friday, December 9
Due Sunday, December 18 at 3pm in AC304 or before in OC332*

Name: __________________________

This test is individual. You may consult your class notes, reference books and your instructor. **Enter all your work and your answers in neat, organized pages and staple them to the test.** Full credit will not be given to confusing, unreadable or disorganized answers. Answers must be derived or explained, not just simply written down.

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Problem 0: Fill the online class evaluation for a few bonus points TBD.

Problem 1: In lecture we analyzed hypothesis testing for the case where a binary random variable $X$ was sent through a channel with additive gaussian noise of zero mean and variance $\sigma^2$ such that random variable $Z = X + N$ was received. Random variable $X$ could be $x_0$ with a priori probability $p$ or 0 with a priori probability $1 - p$. One way of deciding whether a particular message $x_i$ was 0 or $x_0$ was to observe the received value $z_i = x_i + n_i$ and compare it to a fixed parameter $\lambda$. The resulting error probability was

$$P_e = (1 - p) \int_{\lambda}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}} dz + p \int_{-\infty}^{\lambda} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-x_0)^2}{2\sigma^2}} dz$$

Show that

$$\lambda = \frac{\sigma^2}{x_0} \ln \frac{1 - p}{p} + \frac{x_0}{2}$$

minimizes the probability of error $P_e$. What is $\lambda$ if $p = 1/2$? Draw a neat graph of $Z$ to justify the prediction from the equation.

Hint:

$$\frac{d}{dt} \int_{-\infty}^{t} f(\tau)d\tau = f(t)$$
Problem 2: In this problem, you will show that the channel capacity in information bits per transmission bits \((C_S)\) of a binary symmetric memoryless channel with error probability \(p\) is

\[
C_S = 1 + p \log_2 p + (1 - p) \log_2(1 - p)
\]

Recall than in lecture we argued that if \(\beta\) transmission bits are used to encode \(\alpha\) information bits, then, after time \(T\), there will be \(2^{\beta T}\) possible sequences which code \(2^{\alpha T}\) messages. Furthermore, any sequence will contain \(\beta T p\) incorrect bits with probability 1 as \(T \to \infty\). Thus, each of the \(2^{\alpha T}\) messages will be received as all sequences that differ by \(\beta T p\) transmission bits. The number of such sequences per message is

\[
\binom{\beta T}{\beta T p}
\]

Each of the \(2^{\alpha T}\) messages requires \(\binom{\beta T}{\beta T p}\) sequences in order to achieve error-free communication. Thus, the total number of these sequences must be less than the total number of sequences available:

\[
2^{\alpha T} \binom{\beta T}{\beta T p} \leq 2^{\beta T}
\]

Using the Stirling approximation

\[
n! \approx n^n e^{-n} \sqrt{2\pi n}
\]

Find the channel capacity \(C_S\) in information bits per transmission bits by solving for the constraint

\[
\frac{\alpha}{\beta} \leq C_S
\]

where

\[
C_S = 1 + p \log_2 p + (1 - p) \log_2(1 - p)
\]

in the limit when \(T \to \infty\).
Problem 3: What is the entropy of an infinite, discrete source with probabilities

\[ p_i = \frac{1}{2^i} \quad i \in [1, \infty) \]

for each source symbol. How many bits at least do I need to encode this source without any loss of information? Notice that the source has an infinite amount of symbols. Explain your result.

Hint:

\[
\sum_{i=1}^{\infty} \frac{i}{2^i} = \sum_{i=1}^{\infty} \frac{1}{2^i} + \frac{1}{2} \sum_{i=1}^{\infty} \frac{i}{2^i}
\]

Optional: Provide any feedback you feel necessary. What did you like about the class? What didn’t you like? How hard was the class in comparison with the rest of your classes at Olin? Did you feel the level was appropriate? What helped you learn the most? What did you find most interesting?

You have been a wonderful class. I truly enjoyed sharing this experience with you. Thank you. Have a great break! Happy Holidays!