Problem 1:

A. Verify that \( \cos(1.1\pi n) \) aliases to \( \cos(0.9\pi n) \) by creating a plot of \( \cos(1.1\pi t) \) and \( \cos(0.9\pi t) \) for \(-7 \leq t \leq 7\) and then plotting \( \cos(1.1\pi n) \) for \( n \in \{-7, -6, \ldots, 7\} \) on the same axes. Explain clearly this plot.

**Solution:**
Since \( \cos(1.1\pi n) \) aliases to \( \cos(0.9\pi n) \), the underlying graphs of \( \cos(1.1\pi t) \) and \( \cos(0.9\pi t) \) intersect whenever \( t \) is equal to an integer \( n \).

B. Find the transform of \( \cos(\Omega_0 n) \).

**Solution:**

\[
\cos(\Omega_0 n) \iff \sum_{k} \pi \delta(\Omega - \Omega_0 - 2\pi k) + \pi \delta(\Omega + \Omega_0 - 2\pi k), \quad k \in \mathbb{Z}
\]
C. Sketch the transforms of \( \cos(0.9\pi n) \) and \( \cos(1.1\pi n) \) from \(-2\pi\) to \(2\pi\). Explain clearly.

**Solution:**
Both transforms look identical, since the impulses at \(\pm 0.9\pi\) show up at \(\pm 1.1\pi\) in the case of \(\cos(0.9\pi n)\), and the impulses at \(\pm 1.1\pi\) alias down \(\pm 0.9\pi\) in the case of \(\cos(1.1\pi n)\).

Problem 2: The transfer function of the first-order difference equation

\[
y[n] - ay[n - 1] = x[n]
\]

is

\[
H(\Omega) = \frac{1}{1 - ae^{-j\Omega}}
\]

A. Plot the magnitude and phase of \(H(\Omega)\) from \(-3\pi\) to \(3\pi\) when \(a = 0.9\).

**Solution:**
B. Since $e^x \approx 1 + x$ when $x \ll 1,$

$$H_{\text{approx}}(\Omega) = \frac{\frac{1}{a}}{1 - \frac{a}{\Omega} + j\Omega} \approx H(\Omega) = \frac{1}{1 - ae^{-j\Omega}}$$

when $\Omega \approx 2\pi n$. Make a Bode plot of both $H(\Omega)$ and $H_{\text{approx}}(\Omega)$ when $a = 0.9$ and $10^{-3} < \Omega < 2\pi$. What kind of filter is this?

**Solution:**
The filter passes frequencies around 0, so it is a low pass filter.

C. Redo the part A when $a = -0.9$. What kind of filter is this? Explain clearly.

**Solution:**
The filter passes frequencies around $\pi$. This is the highest frequency a discrete system can have (any higher frequencies will be aliased down, as shown in Problem 1), so it is a high pass filter.
D. Find the transfer function for the difference equation below and plot it as in part A. What kind of filter is this? Explain clearly.

\[ y[n] + 0.9y[n - 2] = x[n] \]

**Solution:**

\[ H(\Omega) = \frac{1}{1 + 0.9e^{-2j\Omega}} \]

The filter passes frequencies around \( \pi/2 \) and rejects both frequencies around 0 and \( \pi \). \( \pi/2 \) is the center frequency between the lowest discrete frequency (0) and the highest (\( \pi \)), so it is a band pass filter.

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E. Find the transfer function for the difference equation below and plot it as in part A. This is called a *comb filter*.

\[ y[n] = x[n] - 0.9x[n - 5] \]

**Solution:**

\[ H(\Omega) = 1 - 0.9e^{-5j\Omega} \]
F. Find and sketch the impulse response for the comb filter of part E. *This type of filter is a finite impulse response (FIR) filter. The first order difference equation of part A is an infinite impulse response (IIR) filter.*

**Solution:**

\[
h[n] = \mathcal{F}^{-1}\{H(\Omega)\} = \mathcal{F}^{-1}\{1 - 0.9e^{-5j\Omega}\} = \delta[n] - 0.9\delta[n - 5]
\]
Problem 3:

A. Find and sketch the transfer function \( H_c(j\omega) \) such that

\[
y_c(t) = x_c \left( t - \frac{1}{3f_S} \right)
\]

in the system shown below, assuming \( x_c(t) \) is bandlimited by \( f_{max} \) such that the sampling frequency \( f_S > 2f_{max} \).

\[
x_c(t) \longrightarrow \boxed{C/D} \xrightarrow{x_d[n]=x_c(\frac{n}{f_S})} \boxed{H_d(\Omega)} \xrightarrow{y_d[n]=y_c(\frac{n}{f_S})} \boxed{C/D} \longrightarrow y_c(t)
\]

Solution:

\[
Y_c(j\omega) = X_c(j\omega) e^{-j\frac{\pi}{f_S}}
\]

\[
H_c(j\omega) = \begin{cases} 
e^{-j\frac{\pi}{f_S}} & -2\pi \frac{f_S}{2} \leq \omega \leq 2\pi \frac{f_S}{2} \\ 0 & \text{otherwise} \end{cases}
\]
B. Find and sketch $H_d(\Omega)$.

Solution:

$$H_d(\Omega) = e^{-j\frac{\Omega - 2\pi k}{3}}, \quad -\pi + 2\pi k \leq \Omega \leq \pi + 2\pi k, \quad k \in \mathbb{Z}$$

C. Find $y_d[n]$ in terms of $x_d[n]$.

Solution:

$$Y_d(\Omega) = X_d(\Omega)e^{-j\frac{\Omega}{3}}$$

$$y_d[n] = x_d[n - 1/3]$$

Note that while this statement is technically true, we cannot apply it literally, since $n$ must be an integer. The next two parts give us the actual answer.

D. Assume $x_c(t) = \text{sinc}(\pi f_s t)$. Verify that $x_d[n] = \delta[n]$. Combine both plots in the same set of axes.

Solution:

$$x_d[n] = x_c(n/f_S) = \text{sinc}(\pi f_s n/f_S) = \text{sinc}(\pi n) = \delta[n]$$
E. Find $y_c(t)$ and $y_d[n]$. Explain why $y_d[n] = h_d[n]$. Combine both plots in the same set of axes.

**Solution:**
Since $H_c(j\omega)$ is a delay of $\frac{1}{3f_S}$, then $y_c(t) = \text{sinc} \left[ \pi f_S \left( t - \frac{1}{3f_S} \right) \right]$.

We can invoke again that $y_d[n] = y_c(n/f_S)$ to find $y_d[n]$:

$$y_d[n] = \text{sinc} \left[ \pi \left( n - \frac{1}{3} \right) \right]$$

Regardless of the definition of $x_c(t)$, since $x_d[n] = \delta[n]$, $y_d$ must be the impulse response $h_d[n]$.