Knight's Tours

Ben Hill, Kevin Tostado

December 18, 2004

Abstract

In this paper we discuss the knight's tour, a chess puzzle related to graph theory. We explore the history of the problem, the connection to Hamiltonian paths and circuits, and some techniques for finding the tours and proving their existence. We also delve into magic knight's tours, applications to cryptography, and symmetry in knight's tours.

The Knight's Tour Problem

The knight is the only chess piece that does not move in a straight line. Instead, the legal move for a knight is two spaces in one direction, then one in a perpendicular direction (Figure 1). A knight's tour is a journey around the chessboard in such a way that the knight lands on each square exactly once.

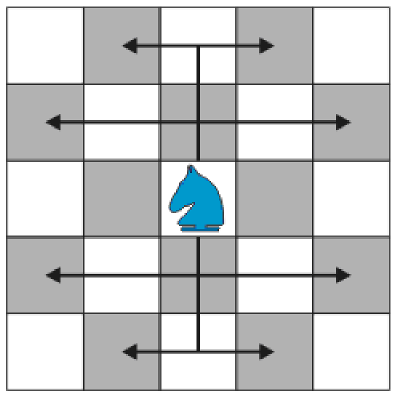

Figure 1: Legal moves for a knight

If you represent a knight's tour as a graph, the vertices would represent the squares of the board and the edges would represent a knight's legal moves between squares (Figure 2). Therefore, a knight's tour (a path that traverses all of the squares of the chessboard) is simply a Hamiltonian path. A closed (or reentrant) knight's tour is one where the final move is one legal move away from the starting square; this is represented by a Hamiltonian circuit.

History of Knight's Tours

The earliest surviving knight's tour is shown in Figure 3. It is attributed to al-Adli ar-Rumi who lived in Baghdad in 840 AD and is known to have written a book on an early form of chess played in the Middle East.

A number of other knight's tours appeared in medieval chess manuscripts. Most of these tours covered only half the board (4x8) or join two such half-

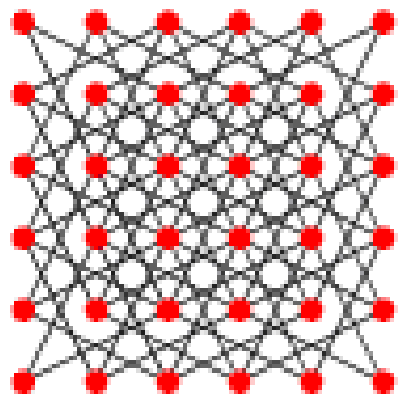

Figure 2: Graph representation of an 8x8 chessboard and knight's moves

boards together. The subject then seems to have been forgotten until around 1722. The first thorough scientific study of knight's tours was conducted by Leonhard Euler in 1759. Euler also investigated symmetries possible within tours [1].

Methods for Finding Knight's Tours

Just as Euler was among the first to study knight's tours systematically, he also created one of the first methods for finding them. Euler's technique (suggested by fellow mathematician L. Bertrand) combines smaller and incomplete tours to form a new tour[1]. The best-known and historically most important procedure for finding knight's tours was invented by German mathematician H.C. Warnsdorff in 1823, and is known as Warnsdorff's rule[2]. The rule is simple: starting at any square, make a legal move to the square having the fewest successors. A successor is any square one legal move away. In using this method the squares with the fewest successors are visited first, which prevents them from becoming dead-ends later on. It is important to note that Warnsdorff's rule is heuristic, meaning it is not guaranteed to find a solution. Although Warnsdorff though his rule would find a tour on any size rectangular board, exhaustive computer searches have shown that it can fail for boards larger than 76x76. A modern day improvement by Arnd Roth breaks ties in choosing successors by selecting the square furthest from

60	11	56	07	54	03	42	01
57	08	59	62	31	64	53	04
12	61	10	55	06	41	02	43
09	58	13	32	63	30	05	52
34	17	36	23	40	27	44	29
37	14	33	20	47	22	51	26
18	35	16	39	24	49	28	45
15	38	19	48	21	46	25	50

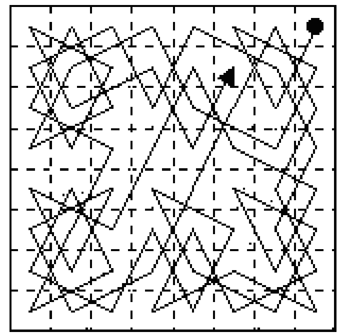

Figure 3: Oldest surviving knight's tour (note it is a closed tour)

the center of the board; this method first fails on a 428x428 board. The reason for using these heuristics instead of an algorithm guaranteed to work is speed. It is possible to create such an algorithm using backtracking, but for larger boards it quickly becomes too slow to be effective (while the heuristics' solution time increases approximately linearly with board size)[3]. Interestingly enough, the most recent methods for finding large knight's tours are somewhat similar to Euler's approach. These techniques decompose a large board into smaller rectangles for which solutions are already known. These smaller solutions are then joined to form a knight's tour[4] (Figure 4).

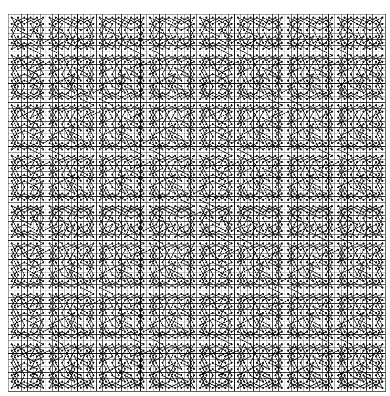

Figure 4: Example of a 60x60 knight's tour generated by decomposition

Existence of Knight's Tours

One interesting knight's tour problem is discovering which boards support a closed tour. Allen Schwenk proposes the following theorem [5], which we will explore in some detail:

Theorem

An m x n chessboard with $m \leq n$ has a knight's tour unless one or more of these three conditions holds:

(i) m and n are both odd;

(ii) $m = 1$, 2, or 4; or

(iii) $m = 3$ and $n = 4$, 6, or 8.

Proof of *(i)*: Looking at the knight's moves, we can see that every legal move is from a black square to a white square, or vice-versa. Thus, any closed tour must visit an equal number of white and black squares. But if both m and n are odd, there are an odd number of squares and the knight cannot an equal number of white and black. Consequently, we cannot construct a closed tour on a board of odd size.

Proof of *(ii)*: If $m = 1$ or 2, it is clear by inspection that the board is not wide enough to construct a tour. When $m = 4$, we prove there cannot be a knight's tour using graph coloring[6]. Two different colorings of a 4x6 board are shown in Figure 5. The important thing to notice about the second board is that there are an equal number of red and blue squares, and from a red square you can only get to a blue square. Putting these facts together means that you can only construct a closed tour if the squares in the sequence of moves alternate between red and blue. From the way the knight moves, we also know that in the first board the knight always alternates between white and black. Since there is no one-to-one correspondence between white/black and red/blue squares, this is a contradiction and there is no possible closed tour.

Proof of *(iii)*: The proof of part *(iii)* involves removing vertices from the Hamiltonian cycle that represents the knight's moves. When k vertices are

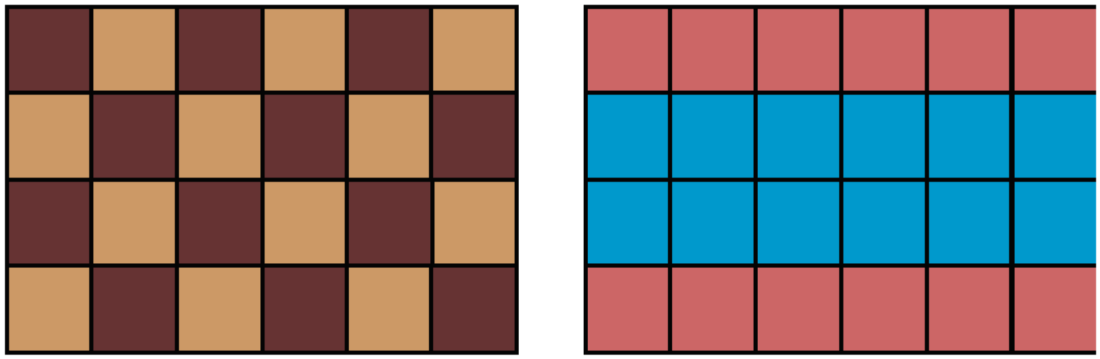

Figure 5: The 4xn board does not support a closed knight's tour

removed from the cycle, there can be at most k connected components of the graph. If there are more than k connected components left after removing k vertices, then the original graph did not contain a Hamiltonian cycle. By proceeding in this manner, it is possible to eliminate the 3x6 and 3x8 boards (the 3x4 board was eliminated in part *(ii)*). In the remainder of the proof, Schwenk goes on to show that it is possible to construct a knight's tour on all other rectangular boards. The proof is too long to include here, but it is a proof by induction with nine base cases (shown in Figure 6). This is similar to the method of constructing knight's tours by decomposing the board into smaller, known tours.

In a similar problem, Watkins and Hoenigman studied closed tours that exist on the surface of a torus, basically a rectangular board in which the knight is allowed to 'wrap around' the board. They found that under this modification, every rectangular board has a tour[7].

Magic Knight's Tours

In a magic knight's tour, the squares of the chess board are numbered in the order of the knight's moves. Each column, row, and diagonal must sum to the same number. The first magic knight's tour (with sum 260) was not found until 1848 by William Beverley. Beverley's tour, however, was actually

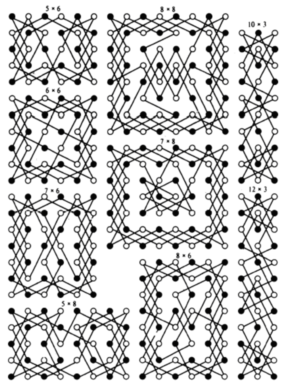

Figure 6: Constituent knight's tours for closed tours on an m x n board

only what is now called semi-magic, because the diagonals did not also add to 260. The tour also exhibits some interesting properties [8](Figure 7).

1	48	31	50	33	16	63	18
30	51	46	3	62	19	14	35
47	2	49	32	15	34	17	64
52	29	4	45	20	61	36	13
5	44	25	56	9	40	21	60
28	53	8	41	24	57	12	37
43	6	55	26	39	10	59	22
54	27	42	7	58	23	38	11

(a)

1	48	31	50	33	16	63	18
30	51	46	3	62	19	14	35
47	2	49	32	15	34	17	64
52	29	4	45	20	61	36	13
5	44	25	56	9	40	21	60
28	53	8	41	24	57	12	37
43	6	55	26	39	10	59	22
54	27	42	7	58	23	38	11

(b)

1	48	31	50	33	16	63	18
30	51	46	3	62	19	14	35
47	2	49	32	15	34	17	64
52	29	4	45	20	61	36	13
5	44	25	56	9	40	21	60
28	53	8	41	24	57	12	37
43	6	55	26	39	10	59	22
54	27	42	7	58	23	38	11

(c)

Figure 7: (a) First semi-magic knight's tour
(b) In each quadrant, the sum of the numbers equals 520 and each of the rows and columns adds to 130
(c) The sum of the numbers in each 2x2 section is 130

Until very recently, the existence of a fully magic knight's tour had yet to be determined. This longstanding open problem has now been settled through the use of exhaustive computer search[9]. After nearly 62 computation-days, the project was completed on August 5, 2003. Although it found a total of 140 semimagic knight's tours, the search showed that no 8x8 fully magic knight's tour is possible[10].

Knight's Tours and Cryptography

A cryptotour is a puzzle in which the 64 words or syllables of a verse are printed on the squares of a chessboard and are to be read in the sequence of a knight's tour. The earliest known examples of a cryptotour were printed in the mid 1800s in a French magazine. In the last few years (1870-1874) of Howard Stanton's chess column in Illustrated London News, he featured 16 cryptotours. Published before the invention of crossword puzzles, these were very popular. Most of the cryptotours featured well known verse from such writers as Dante, Shakespeare, and Walter Scott. Figure 8 gives an example of one of those cryptotours[1].

```
PUZZLE
sor    to     king   good   say    luck   loy    eth
and    moth   a      soon   dis    our    to     bad
place  ry     church his    force  is     hat    al
er     queen  him    wight  he     to     may    truth
man    his    and    and    chess  es     knight op's
a      sneer  the    and    un     lawn   of     tates
cas    that   at     less   pawn   no     bish   lant
eth    faith  tles   hath   the    gal    in     love

TOUR SOLUTION               VERSE SOLUTION
14 55 22 37 12 51 18 35    The man that have no love of chess
23 38 13 54 17 36 11 50    Is, truth to say, a sorry wight,
56 15 40 21 52  9 34 19    Disloyal to his king and queen.
39 24 53 16 33 20 49 10    A faithless and ungallant knight;
 2 57 28 41  8 61 32 47    He hateth our good mother church,
25 42  1 60 29 48  7 62    And sneereth at the bishop's lawn;
58  3 44 27 64  5 46 31    May bad luck force him soon to place
43 26 59  4 45 30 63  6    His castles and estates in pawn!
```

Figure 8: Example of a cryptotour from 1870

Knight's Tours and Symmetry

Using his method for constructing knight's tours, Euler studied some special types of tours. He constructed symmetric tours, having the property that the numbers in diametrically opposite squares have a constant difference of 32. This symmetry was visible geometrically in the Hamiltonian graphs of the knight's moves with the last connected to the first, resulting in a geometrical closed tour[1].

Euler also gave the first ever examples of tours exhibiting quaternary symmetry. These consist of two tours of cross-shaped boards symmetric by reflection about the two diagonals, and a 10 x 10 tour formed by repeating the same 5x5 tour in each quarter, rotated 90[1].

Knight's tours can also be used to construct tessellations. Using the four sets of 16 moves shown in Figure 9, it is possible to construct four polygons that together can form a tessellation. Knight's tour tessellations can even be used to create beautiful 3-D patterns such as astersphaira (star spheres) and hexastersphaira (Figure 10)[11].

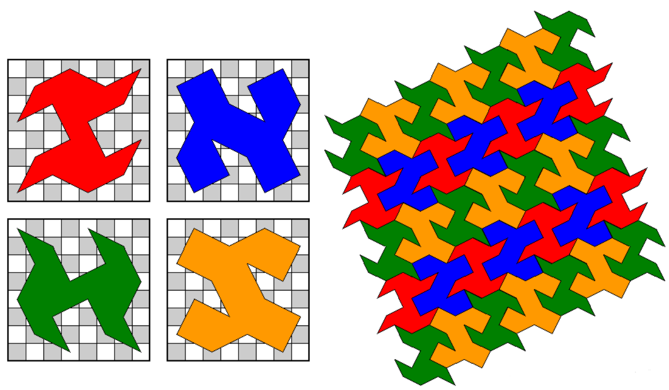

Figure 9: Four polygons formed by knight moves and their tessellation of the plane

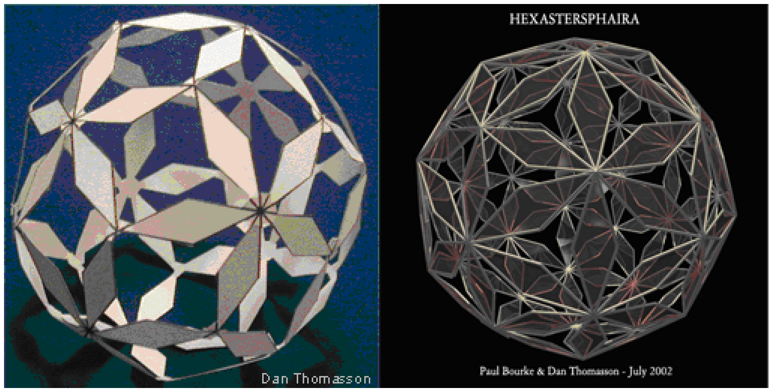

Figure 10: Astersphaira (left) and hexastersphaira(right)

References

[1] Jelliss, George. "Introducing Knight's Tours." http://homepages.stayfree.co.uk/gpj/ktn.htm

[2] Trnberg, Gunno. "Warnsdorff's Rule." http://web.telia.com/ u85905224/knight/eWarnsd.htm

[3] Roth, Arnd. "The Problem of the Knight." http://sun0.mpimf-heidelberg.mpg.de/people/roth/Mma/Knight.html

[4] Parberry, Ian. "Scalability of a Neural Network for the Knight's Tour Problem." (1997)

[5] Schwenk, Allen J. "Which Rectangular Chessboards Have a Knight's Tour?". Mathematics. 64 (1991): 325-332.

[6] Stewart, Ian. "Mathematical Recreations: Knight's Tours." Scientific American. April 1997. 102-104.

[7] Watkins, John J. and Rebecca L. Hoenigman. "Knight's Tours on a Torus." Mathematics. 70 (1997): 175-184

[8] Collins, Edward D. "A Knight's Tour." http://www.edcollins.com/chess/knights-tour.htm

[9] Stertenbrink, Guenter. http://magictour.free.fr

[10] Weisstein, Eric W. "There Are No Magic Knight's Tours on the Chessboard." http://mathworld.wolfram.com/news/2003-08-06/magictours/

[11] Thomasson, Dan. "The Knight's Tour." http://www.borderschess.org/KnightTour.htm