Exercise 3: Solutions

Transport in Biological Systems

Fall 2015

1. Modeling blood flow in a full scale model ($D_M = D_V$) with a model fluid. Given the parameters listed below, what are the flow and pulse rates to use?

parameter	in vivo (V)	model (M)
ν (cs)	4	6
ρ (g/cc)	1	1
Q (cc/s)	15	?
ω (min^{-1})	65	?
τ ($dynes/cm^2$)	?	20

   (a) We need to match Reynolds numbers ($Re_M = Re_V$), from which we can get the appropriate flow rate in the model, Q_M. We know Q_V, so given that $v = \frac{Q}{A}$ and $Re = \frac{vD}{\nu}$, we can say that $Q_M = Q_V \frac{\nu_M}{\nu_V} = (15)(\frac{6}{4}) = 22.5$ cc/s (or ml/s). NOTE: You are given a volumetric flow rate so need to substitute $v = \frac{Q}{A}$ into Re to get the units right (i.e. velocity is not in ml/s).

   (b) Since we need to match the pulse rate, we need to maintain the Womersley number (you could also use the Strouhal number), $\alpha = \sqrt{\frac{\omega R^2}{\nu}}$ for the two systems. Thus, $\omega_M = \omega_V \frac{\nu_M}{\nu_V} = 97.5 min^{-1}$. So in both cases, the kinematic viscosity is key!

   (c) To consider shear stresses, recall that for flow in a tube, $\tau = \frac{4Q\mu}{\pi R^3}$.

      i. Noting that $1g = 1 dyne.s^2/cm$, $1 poise = 1\ dyne.s/cm^2$, and $1 stoke = 1\ cm^2/s$, we can do some unit converstion to say that $\rho = 1 dyne.s^2/cm^4$.

      ii. Then we can get that $\mu_V = \nu_V \rho_V = 0.04\ dyn.s/cm^2$, and $\mu_M = 0.06\ dyn.s/cm^2$.

      iii. But we need D! Solving with the parameters we have, $D = 0.88$ cm (8.8 mm is capillary-sized, so small but reasonable).

      iv. So finally we can solve for shear in the in vivo system, $\tau_V = 9\ dyne/cm^2$. Pretty resonable number! Check. But interesting that similarity doesn't maintain shear, right?

      v. Note that approaches using the Fanning friction factor or making arguements about shear rate ($\frac{dv}{dr}$) scaling with v for equivalent D also give the same answer.

2. In this case, the diameter of the model is 4 times that of the in vivo system and a fixed pump speed (see table below). Units were people's downfall here.

parameter	in vivo (V)	model (M)
D (cm)	0.3	1.2
$\omega(s^{-1})$	?	0.33
Re	90	90
α	3.1	3.1

(a) First, we are asked to get the kinematic viscosity of the working fluid, ν_M, which we can do from the Womersley number, $\nu_M = \frac{\omega_M R_M^2}{\alpha^2} = 0.012 cm^2/s$, or 1.2 cs. Note that if you don't use R in cm, you get a different number.

(b) Next, we want the inlet velocities of the two systems. We know that $Re = \frac{vD}{\nu} = 90$. Thus, for the model, $v_M = \frac{\nu_M Re}{D_M} = 0.94 cm/s$.

(c) What about in vivo values? We actually need some extra information here. If we use $\nu_V = 4cs = 0.04\ cm^2/s$ from the first problem, $v_V = 12\ cm/s$. This is much higher than what we calculated for the model system, which is bigger. Interesting.

(d) Further, using $\nu_V = 0.04 cm^2/s$ and $\alpha = 3.1$, we can estimate $\omega_V = 17\ s^{-1}$, which would give us a period of 0.06s for pulsatility in thee body? This doesn't seem right.

(e) A different way to approach this is to say that 1 Hz is a good guess for pulse rate in vivo. Using the Womersley number to calculate the kinematic viscosity, $\nu_V = 0.0023\ cm^2/s$. This also doesn't seem right, but would give $v_V = 0.7 cm/s$ (a more reasonable number).

(f) What can we conclude? First off, we needed another piece of information that wasn't given in the problem to calculate in vivo values. Second, using things we know make sense, we get very strange values back for other parameters. Thus, the best we can conclude is that we won't be able to match in vivo values using the given parameters. For an in vivo system with a Womersley number of 3.1, a frequency of 1 s^{-1}, and a kinematic viscosity of 0.04 cm^2/s, we would need a radius of $R_V = \alpha\sqrt{\frac{\nu}{\omega}} = 0.62$ cm, or a diameter of 1.24 cm... so more like the model!

3. For this problem, we have 2 parallel plates, with the top one moving at a speed, U. The plates can be considered infinite in length and width (i.e. in x and z) relative to the gap between them (in y), h. There is also a pressure drop across the plates in the x direction, ΔP.

   (a) In considering the Navier-Stokes equations, we're using cartesian coordinates and all we need is the x component:

$$\rho(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}) = -\frac{\partial P}{\partial x} + \mu(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}) + \rho g_x$$

   Based on our previous analysis, we know that $v_y = v_z = 0$ and $v_x = f(y)$. Also, flow is steady. These eliminate all terms on the left and all but 2 terms on the right, giving:

$$\frac{dP}{dx} = \mu\frac{d^2 v_x}{dy^2}$$

   (b) As we have argued before, because the LHS is a function of x and the RHS is a function of y, they must be equal to a constant. Also, as we have invoked before, $\frac{dP}{dx} = -\frac{\Delta P}{L}$, where L is the length of the plates. Then,

$$\mu\frac{dv_x}{dy} = -\frac{\Delta P}{L}y + C_1$$
$$v_x(y) = -\frac{\Delta P}{2\mu L}y^2 + \frac{C_1}{\mu}y + C_2$$

(c) Given the no slip boundary conditions on both plates, $v_x(0) = 0$ and $v_x(h) = U$, we can say:

$$C_2 = 0$$
$$C_1 = \frac{U\mu}{h} + \frac{\Delta P h}{2L}$$
$$v_x(y) = \frac{\Delta P}{2\mu L}(y(h-y)) + \frac{U}{h}y$$

Note that the second term in this expression is the original velocity profile we got for Couette flow.

(d) The shear stress is then:

$$\tau = \mu\frac{dv_x}{dy} = \frac{\Delta P h}{2L} - \frac{\Delta P y}{L} + \mu\frac{U}{h}$$

Letting τ_0 and τ_h be the wall shear stress evaluated at the top (y=0) and bottom (y=h) plates, respectively:

$$\tau_0 = \frac{\Delta P h}{2L} + \mu\frac{U}{h}$$
$$\tau_h = -\frac{\Delta P h}{L} + \mu\frac{U}{h}$$

(e) Assuming the width of the plates is w, we can calculate the average velocity and flow rate:

$$< v_x >= \frac{1}{wh}\int_0^h \int_0^w v_x dz dy = \frac{\Delta P h^2}{12\mu L} + \frac{U}{2}$$
$$Q = A < v_x >= \frac{\Delta P w h^3}{12\mu L} + \frac{Uwh}{2}$$

(f) Using this, we can calculate that for $-\frac{U}{\frac{\Delta P}{L}} = \frac{h^2}{6\mu}$, average velocity (and flow rate) are zero.

(g) The figure below shows the velocity profiles for different values of $\frac{h^2}{6\mu}$. Note that for $U >> -\frac{\Delta P}{L}$ the profile looks linear, like the plate moving example we did in class (Couette flow), while for $U << -\frac{\Delta P}{L}$ the profile looks parabolic (like Poiseuille flow) but with different velocities at the boundaries.

4. In my book, it says $1/Sr$ in front of the time-dependent term in the de-dimensionalized Navier-Stokes. It should be just Sr.

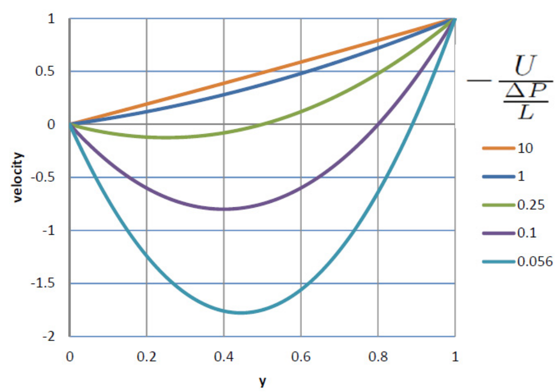

Figure 1: Velocity profiles for different ratios of plate velocity, U to $\frac{\Delta P}{L}$.