

Exercise 5: Solutions

Transport in Biological Systems

Fall 2015

1. We know from previous examples that the velocity profile in a cylinder is

$$v_z = \frac{\Delta P}{4\mu L}(R^2 - r^2).$$

So the shear stress, evaluated at the wall, is

$$\tau_{rz} = \mu \frac{dv_z}{dr} = -\frac{\Delta P r}{2L} \Big|_R = -\frac{\Delta P R}{2L}.$$

- (a) And then taking the Poiseuille equation, $Q = \Delta P \pi R^4 / 8\mu L$, we can express shear stress as

$$\tau = \frac{4Q\mu}{\pi R^3}.$$

- (b) For a mean velocity $\langle v_z \rangle = 15 \text{ cm/s}$, a diameter of $D=1 \text{ cm}$, a viscosity of $\mu = 4\text{cp} = 0.04 \text{ dyn} \cdot \text{s/cm}^2$, the area of the vessel is 0.79 cm^2 , giving a flowrate of $Q = A \langle v_z \rangle = 11.8 \text{ cm}^3/\text{s}$ and a wall shear stress in the abdominal aorta is $\tau = 4.8 \text{ dynes/cm}^2$.
2. In many ways this is the same set up that we have seen. By inspection of the system, we can say that $v_y = v_z = 0$ and $v_x(y)$ only. We assume steady state and have no pressure gradient.
 - (a) Thus, as we like to do, we can start with only the x-component of the Navier-Stokes equation:

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x.$$

We can eliminate the first term because there is no time dependence, second term because v_x is only a function of y , and the third and fourth terms on the left hand side because $v_y = v_z = 0$. On the right hand side, we are told there is no pressure drop, we will neglect gravity, and invoke that v_x is only a function of y again to be left with a simple expression which we can integrate, using A and B and integration constants:

$$\begin{aligned} \frac{d^2 v_x}{dy^2} &= 0 \\ v_x &= Ay + B \end{aligned}$$

The boundary conditions are $v_x(0) = 0$ and $v_x(h) = U$. Note that things work out even if you don't choose this coordinate system but this one is very handy. Applying these we find that

$$\begin{aligned} B &= 0 \\ A &= \frac{U}{h} \\ v_x &= \frac{Uy}{h} \\ \tau_{yx} &= \mu \frac{dv_x}{dy} = \frac{\mu U}{h} \end{aligned}$$

- (b) Graphing these shows that shear stress is constant in y and that v is a straight line in y .
(c) We know that $Q = A \langle v \rangle$ so we can find average velocity and Q :

$$\begin{aligned} \langle v_x \rangle &= \frac{1}{wh} \int_0^h \int_0^w v_x dz dy = \frac{Uy^2}{2h^2} \Big|_0^h = \frac{U}{2} \\ Q &= \frac{whU}{2} \end{aligned}$$

- (d) What forces are acting here? No body forces, only surface forces, and really only shear. The force required to move the plate is simply $\tau_{yx} = \frac{\mu U}{h}$, a force per unit area.
(e) This is a good setup to measure viscosity because if you know the speed of the plate, the force required to move it, and the distance between plates, and the area of the plate, let's say it is Lw (letting L be the length in x), then $\tau = F/Lw$ and $\mu = Fh/ULw$, so you get viscosity from a few simple measurements and the geometry of the system!
3. Now we have an oscillating pressure gradient in the x direction for our friends, the parallel plates.
- (a) If we assume that flow is laminar, we should still not have flows in the y and z direction, so like before, we can say that $v_y = v_z = 0$. We can also say that v_x is not a function of z , but it is going to be a function of y , giving us the velocity profile, and t because of the time-dependence. x is a little more complicated. For rapidly oscillating pressure, the velocity profile will vary in x as the fluid "catches up" with the change, but for relatively slowly oscillating pressure the velocity profile will "instantly" change with the pressure, and there will be no x -dependence of velocity.
(b) Either way, we only need the x component of the Navier-Stokes equation:

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x.$$

where, for slow oscillations, the second through fourth terms on the left are zero as described above, as are the second and fourth on the right. Again we neglect gravity. This leaves:

$$\rho \frac{\partial v_x(y, t)}{\partial t} = -\frac{\partial P(x, t)}{\partial x} + \mu \frac{\partial^2 v_x(y, t)}{\partial y^2}.$$

where P is a function of x and t as well, though as we have done before, we can likely argue that $\Delta P(t)/L$ is an equivalent term and that ΔP is not dependent on x to remove the x dependence...

- (c) The boundary conditions, since we have stationary plates are $v_x(0) = v_x(h) = 0$.
- (d) Substituting for pressure, we can say that:

$$\rho \frac{\partial v_x(y, t)}{\partial t} + \frac{\Delta P(t)}{L} = \mu \frac{\partial^2 v_x(y, t)}{\partial y^2},$$

which math folk would call the heat equation with a nonlinear term... and I'm not going to try to solve but please let me know if you do.

- (e) Assuming rapid oscillation, we still only need the x component of the Navier-Stokes equation:

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x.$$

but now we have to keep the $v(x)$ terms to get:

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} \right) = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right)$$

with of course the same boundary conditions, just NOW! more complicated to solve!

4. Here we consider a muscle in a well stirred bath and want to make sure it is not hypoxic (too low oxygen).
- (a) We start by developing an expression describing the concentration of oxygen in the cylindrical muscle, radius R_{adius} (because there are too many R's in this problem and I'm a big dork), with reaction (described by R_{O_2}) at steady state:

$$\begin{aligned} 0 &= \frac{D}{r} \frac{d}{dr} \left(r \frac{dC}{dr} \right) - R_{O_2} \\ C(R_{adius}) &= C_{bath} \\ N \Big|_{r=0} &= -D \frac{dC}{dr} \Big|_{r=0} = 0 \end{aligned}$$

Where C_{bath} is the concentration of the bath (see below). We can then integrate and use A and B as integration constants to get:

$$\begin{aligned} \frac{dC}{dr} &= \frac{R_{O_2}}{2D} r + \frac{A}{r} \\ C(r) &= \frac{R_{O_2} r^2}{4D} + A \ln(r) + B \end{aligned}$$

Applying the boundary conditions gives us $A = 0$ and $B = C_{bath} - \frac{R_{O_2} R_{adius}^2}{4D}$ so we can say that:

$$C(r) = C_{bath} - \frac{R_{O_2}}{4D} (R_{adius}^2 - r^2)$$

Since we're ultimately interested in the lowest concentration in the center, we evaluate at $r=0$:

$$C(0) = C_{middle} = C_{bath} - \frac{R_{O_2}}{4D} R_{adius}^2$$

$$R_{adius} = \sqrt{\frac{4D(C_{bath} - C_{middle})}{R_{O_2}}}$$

- (b) To get to calculating some values, we need to do a couple of things. We are given a bunch of parameters and we need to convert them and also make some assumptions (all summarized in the table at the end). It's important to choose a basis for units; I'm choosing to do things in moles, mL, mmHg, and seconds.
- (c) First, we can use Henry's law, $P_{O_2} = k_H C$, to convert the partial pressure of oxygen in the bath to a concentration, where k_H is the Henry's law constant. (Recall from Chemistry that Henry's law relates the partial pressure of dissolved gases to the concentration.) We look up value for $k_H = 769.2 \text{ L} \cdot \text{atm/mol}$ for oxygen in water at 25°C , which is probably close enough. Converting that to our base units, this is $5.85 \times 10^8 \text{ mL} \cdot \text{mmHg/mol}$. We can use the value of $P_{O_2} = 600 \text{ mmHg}$ in our well-mixed saline to calculate the oxygen concentration in the bath, $C_{bath} = 1.03 \times 10^{-6} \text{ mol/mL}$.
- (d) Next, let's find the minimum concentration we can have, which will be in the middle. Since it's not a liquid exactly, we need the solubility, which is sort of analogous to the Henry's law constant. Converting solubility to our base units gives us, $S_{tissue} = 3.16 \times 10^{-5} \text{ ml oxygen / ml tissue} \cdot \text{mmHg}$. We can multiply $S_{tissue} P_{min,tissue}$, but that gives us ml oxygen / ml tissue and we need moles oxygen / ml tissue. For lack of a better idea, we can invoke the ideal gas law and an atmospheric pressure of 760 mmHg, $\frac{n_{O_2}}{V} = \frac{P}{R_{gas}T}$. Then, $S_{tissue} = 1.24 \times 10^{-9} \text{ mol oxygen / ml tissue} \cdot \text{mmHg}$ and $C_{middle} = S_{tissue} P_{min,tissue} = 2.48 \times 10^{-8} \text{ mol oxygen/mL tissue}$.
- (e) Then we can tackle the consumption rate, using the highest value (10 mL O_2 / 100 g tissue·min) as the worst case scenario, invoking our friend the ideal gas law again, and assuming a tissue density of 1 g/mL (this is, in fact, reasonable), we can calculate the reaction rate of oxygen as $R_{O_2} = 6.55 \times 10^{-8} \text{ mol/mL} \cdot \text{s}$.
- (f) Finally (!) we can calculate R_{adius} using all the numbers we have. For the worse case scenario ($D = 1 \times 10^{-5}$ and consumption of 10 ml oxygen/100 g tissue·min), $R_{adius} = 0.025 \text{ cm}$ (or 250 μm) and for the best case scenario ($D = 3 \times 10^{-5}$ and consumption of 2 ml oxygen/100 g tissue·min), $R_{adius} = 0.096 \text{ cm}$ (or 960 μm or about 1 mm). This is pretty tinsy and would be ridiculously hard to isolate but would be doable by a very good physiologist.

| Parameter | Value | Units | Source |
|------------------|------------------------|------------------------------|--------------------------|
| P_{O_2bath} | 600 | mmHg | given |
| C_{bath} | 1.03×10^{-6} | mol/mL | calculated |
| $P_{tissue,min}$ | 20 | mmHg | given |
| $C_{min,tissue}$ | 2.48×10^{-8} | mol/mL | calculated |
| S_{tissue} | 1.24×10^{-9} | mol oxygen / mL tissue·mmHg | converted |
| consumption | 2-10 | mL oxygen / 100 g tissue·min | given |
| R_{O_2} | 6.55×10^{-8} | mol/mL·s | converted and calculated |
| D | $1 - 3 \times 10^{-5}$ | cm^2/s | given |
| T | 310 | K | assumed |
| ρ | 1 | g/mL | assumed |
| k_H | 5.85×10^8 | mL·mmHg/mol | looked up |
| R_{gas} | 62.36367 | L·mmHg/K·mol | assumed |
| P | 760 | mmHg | assumed |