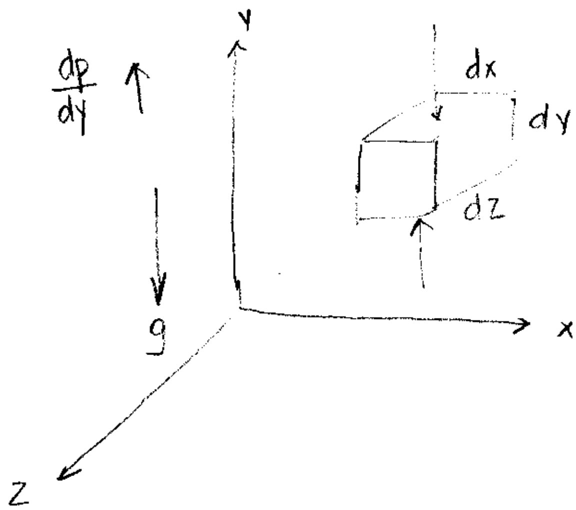
Fluid Statics

Outline

- Hydrostatic equation
- Hydrostatic forces on surfaces
- Buoyancy

Hydrostatic Equation



Fluid element at rest, $\Sigma F = 0$

Pressure force + gravity force = 0

$$p\,dz\,dx - \left(p + \frac{dp}{dy}dy\right)dz\,dx - \rho g\,dx\,dy\,dz = 0$$

$$-\frac{dp}{dy}dy\,dA - \rho g\,d\forall = 0$$

$$-\frac{dp}{dy} - \rho g = 0$$

$$\boxed{dp = -\rho g\,dy}$$ Diff. form of hydrostatic eq.

For constant ρ, integrate

$$p(y) = p_o - \rho g y$$

$$p_o = p(y=0)$$

$$-\nabla p - \rho\vec{g} = \rho\vec{a}$$

∇p = pressure gradient in vector form

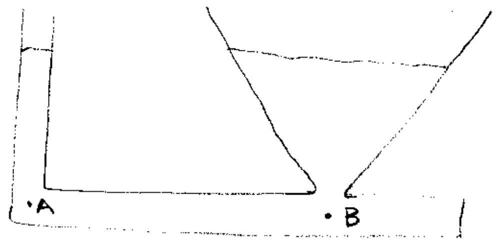
$$\nabla p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$$

- At rest or moving with rigid body motion
- No shearing stresses

absolute, gage, vacuum pressure

Two situations

①



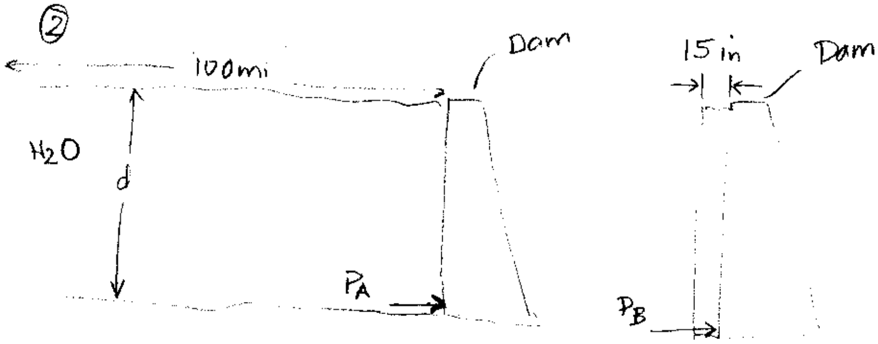
$P_A > P_B$

$P_A < P_B$?

$P_A = P_B$

→ and with Hg

②



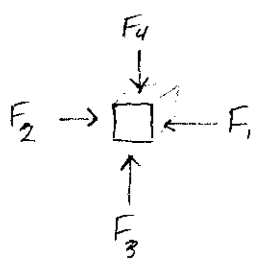
Horizontal thrust on dam

$P_A > P_B$

$P_A < P_B$?

$P_A = P_B$

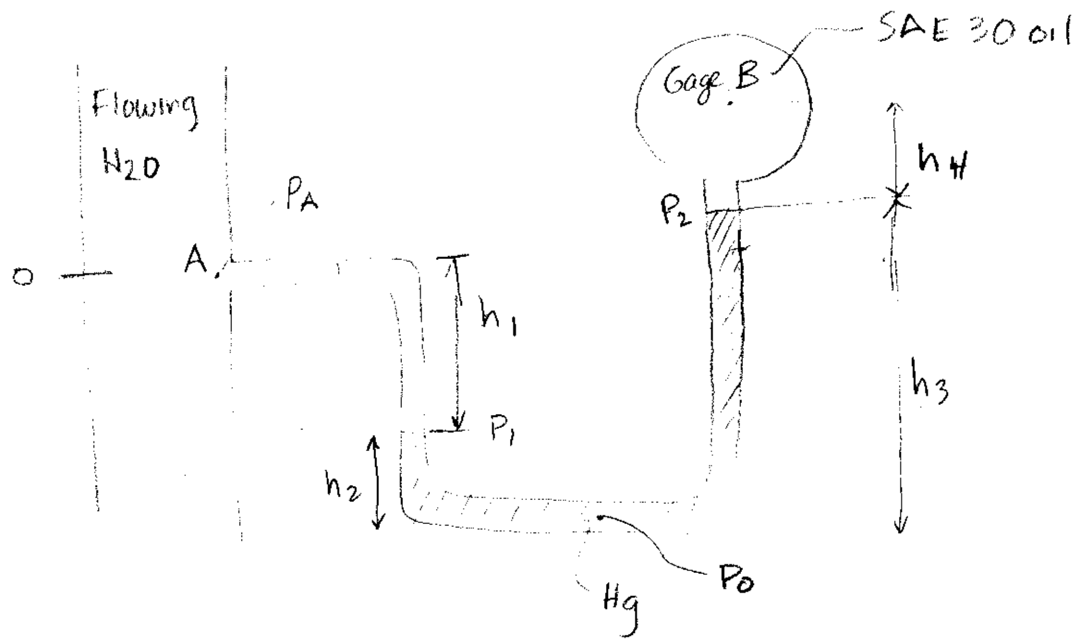
Hydrostatic pressure at a point



• No motion → no shear stresses

$F_1 = F_2$

as $d\forall \rightarrow 0$ $F_3 = F_4$



ρ_{H_2O}, ρ_{Hg}, ρ_{oil}

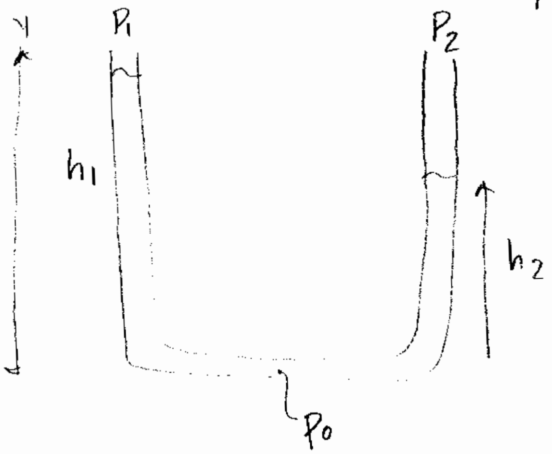
$$P_A - P_B = (P_A - P_1) + (P_1 - P_2) + (P_2 - P_B)$$

$$P_A - P_B = -\rho_{H_2O}\, g\, h_1 + \rho_{Hg}\, g (h_3 - h_2) + \rho_{oil}\, g (h_4)$$

up - down

jump across

Example: Manometer (pressure meas. device)

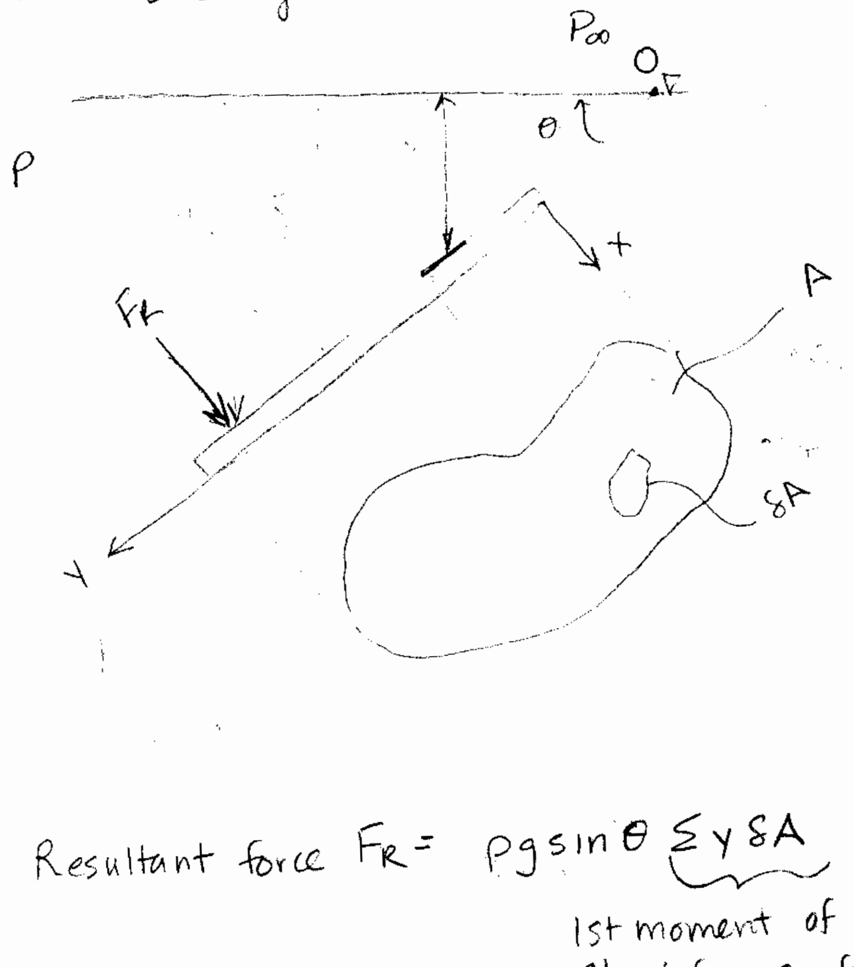


$$p_1 = p_0 - \rho g h_1$$

$$p_2 = p_0 - \rho g h_2$$

$$p_2 - p_1 = \rho g (h_1 - h_2)$$

Plate submerged in a fluid



Force on δA is

$$F = p\,\delta A$$

$$F = \rho g\, y \sin\theta\, \delta A$$

Resultant force $F_R = \rho g \sin\theta \underbrace{\sum y\,\delta A}_{\text{1st moment of area about free surface}}$

$F_R = \rho g \sin\theta\, A\bar{y}$ area → centroid → magnitude

location of resultant force → center of pressure

To find center of pressure, ΣM_x

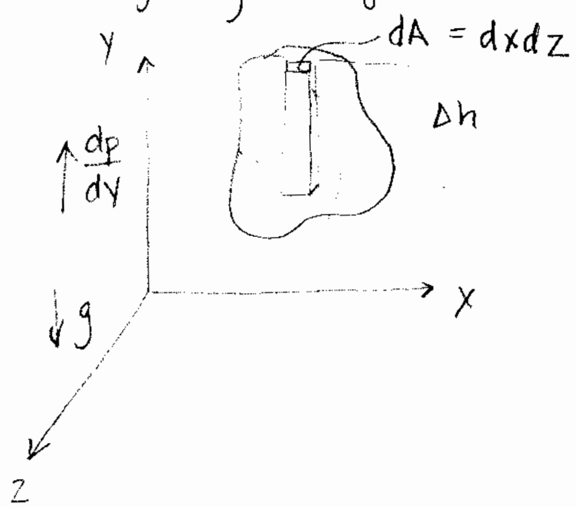
$$F_R y_R = \Sigma y F$$
$$= \Sigma y (\rho g y \sin\theta \, \delta A)$$
$$= \rho g \sin\theta \underbrace{\Sigma y^2 \delta A}_{\text{2nd moment of inertia, } I_x}$$

$$y_R = \frac{\cancel{\rho g \sin\theta} \, I_x}{\cancel{\rho g \sin\theta} \, A \bar{y}} = \frac{I_x}{A\bar{y}}$$

Parallel axis theorem

$$y_R = \frac{I_{xc}}{y_c A} + y_c$$

Buoyancy - object immersed in fluid



$$dF_y = p\,dA - \left(p + \frac{dp}{dy}\Delta h\right) dA$$
$$= \frac{-dp}{dy} \Delta h \, dA$$
$$dF_y = \rho g \, d\forall$$

$$F_y = \rho g \forall \quad \text{buoyancy force}$$

fluid (ρ), object ($\forall$)

Archimedes: Buoyancy force = weight displaced by body

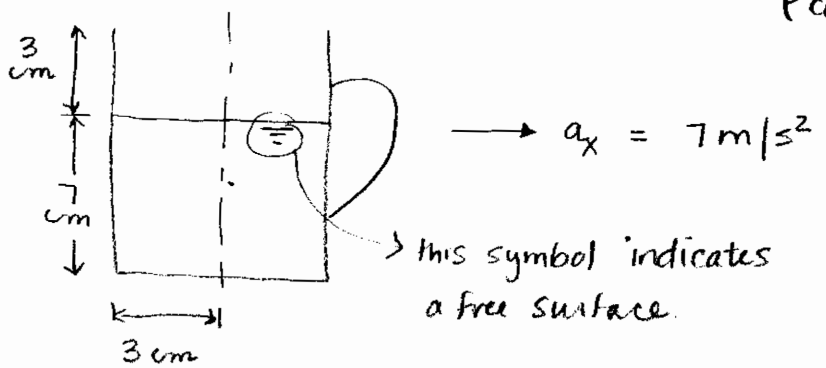
Concept question: Soda bottle video V2.5 movie

Pressure distribution in rigid body motion

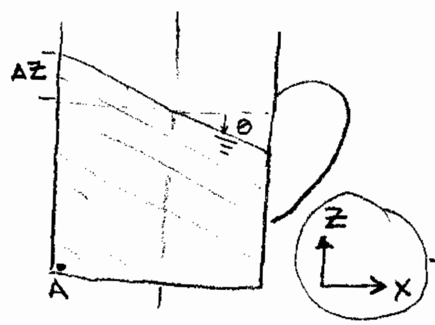
Coffee mug in horizontal cup holder while I'm drag-racing my VW Jetta.

Constant acceleration of 7 m/s²

$\rho_{coffee} = 1010 \ kg/m^3$



i) Draw free surface at steady state and indicate lines of constant pressure. (Qualitatively)



Always show your coordinate system!

ii) Will liquid spill out of the mug?

Hydrostatic Equation

$(\nabla p = \rho(\vec{g} - \vec{a}))$

$$-\nabla p + \rho \vec{g} = \rho \vec{a}$$

$$-\frac{\partial p}{\partial x} + \cancel{\rho g_x}^{0} = \rho a_x$$

$$-\frac{\partial p}{\partial z} + \rho g_z = \cancel{\rho a_z}^{0}$$

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial z} dz$$

$$dp = -\rho a_x dx + \rho g_z dz$$

Pick a line of constant pressure $(dp = 0)$ and find slope $\left(\frac{dz}{dx} = ?\right)$

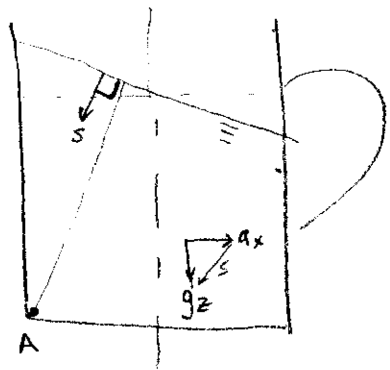
$$\frac{dz}{dx} = \frac{a_x}{g_z}$$

$$\frac{\Delta z}{\Delta x} = \tan\theta = \frac{a_x}{g_z}$$

$$\theta = \tan^{-1}\left(\frac{7 m/s^2}{9.81 m/s^2}\right) = 35.5°$$

$$\Delta z = (3cm)(\tan 35.5) = 2.14 \text{ cm} \quad \text{(won't spill)}$$

ii) What is the pressure at point A?



$$\frac{dp}{ds} = \rho g_s$$

$$g_s = (a_x^2 + g_z^2)^{1/2}$$

$$P_A = \Delta s \, \rho \, (a_x^2 + g_z^2)^{1/2}$$

$$= ((0.07 + 0.0214)\cos 35.5°)(1010 \text{ kg/m}^3)((7 m/s)^2 + (9.81 m/s)^2)^{1/2}$$

$$= 906 \text{ Pa}$$

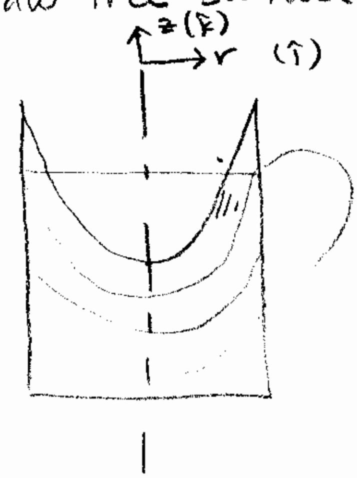
General Equations

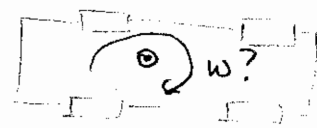
$$\nabla p = \rho(\vec{g} - \vec{a})$$

$$\theta = \tan^{-1} \frac{a_x}{g_z + a_z}$$

The drag racer goes into an extended spin, with the center of rotation coincidentally around the center axis of the coffee mug. We notice that the coffee just reaches the lip of the mug. At what angular velocity is the car spinning?

i) Draw free surface and lines of constant pressure





ii) Find ω

$$\vec{V} = \vec{\omega} \times \vec{r}$$

$$\vec{a} = \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{centripetal acceleration}} + \cancel{\frac{d\vec{\omega}}{dt} \times \vec{r}}$$

constant angular velocity

$$\vec{\omega} = \omega \hat{k} \qquad \vec{r} = r\hat{i}$$

$$\vec{a} = -r\omega^2 \hat{i}$$

$$\nabla p = \rho(\vec{g} - \vec{a})$$

$$\nabla p = \rho(-g\hat{k} + r\omega^2 \hat{i})$$

$$\frac{\partial p}{\partial z} = -\rho g \qquad \frac{\partial p}{\partial r} = \rho r \omega^2$$

$$dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial z} dz$$

$$dp = \rho r \omega^2 dr - \rho g dz$$

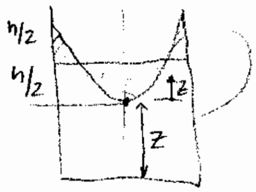
$dp = 0$, $\frac{dz}{dr} = ?$

$$\frac{dz}{dr} = \frac{r\omega^2}{g}$$

$$z = \frac{r^2\omega^2}{2g} + \text{constant}$$

Need a Known height at a Known r, ω

$r = 0, \quad \omega = 0, \quad z = 0$



$$\frac{h}{2} = \frac{r^2 \omega^2}{4g}$$

$\omega = 36.2$ rad/s $= 345$ rev/min