Franklin W. Olin College of Engineering
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# Review of Vector Calculus

## Gradient of a scalar

If P is a scalar function of x, y, and z:

$$P = P(x, y, z)$$

Then the gradient of P (the pressure gradient) is a vector quantity of the form:

$$\nabla P = \frac{\partial P}{\partial x}\hat{i} + \frac{\partial P}{\partial y}\hat{j} + \frac{\partial P}{\partial z}\hat{k}$$

The gradient of P is often also referred to as "del" P.

## Divergence of a vector

If the velocity vector $\vec{V}$ has velocity components of u, v, and w – and each velocity component is a function of x, y, and z:

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$
$$u = u(x, y, z)$$
$$v = v(x, y, z)$$
$$w = w(x, y, z)$$

Then the divergence of the velocity is a scalar quantity of the form:

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The reverse of the scalar product $\nabla \cdot \vec{V}$ is $\vec{V} \cdot \nabla$ and is a scalar differential operator:

$$\vec{V} \cdot \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

It is common to enclose this operator in parentheses. Examples of using this operator on a scalar (density) and a vector (velocity) are given below:

$$\left(\vec{V}\cdot\nabla\right)\rho = u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z} \text{ (scalar quantity)}$$

$$\left(\vec{V}\cdot\nabla\right)\vec{V} = u\frac{\partial\vec{V}}{\partial x}+v\frac{\partial\vec{V}}{\partial y}+w\frac{\partial\vec{V}}{\partial z} \text{ (vector quantity)}$$

The vector quantity $\left(\vec{V}\cdot\nabla\right)\vec{V}$ can be expanded as follows:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} \text{ (x-component)}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} \text{ (y-component)}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} \text{ (z-component)}$$

Laplacian of a scalar or vector

The Laplacian operator can operate on a scalar or a vector and is given by the symbol $\nabla^2$.

$$\nabla^2\rho = \frac{\partial^2\rho}{\partial x^2}+\frac{\partial^2\rho}{\partial y^2}+\frac{\partial^2\rho}{\partial z^2} \text{ (scalar quantity)}$$

$$\nabla^2\vec{V} = \frac{\partial^2\vec{V}}{\partial x^2}+\frac{\partial^2\vec{V}}{\partial y^2}+\frac{\partial^2\vec{V}}{\partial z^2} \text{ (vector quantity)}$$

Curl of a vector

The curl of a vector $\vec{V}$ is given by the operator $\nabla\times\vec{V}$.

$$\nabla\times\vec{V} = \left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\hat{i}+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\hat{j}+\left(\frac{\partial v}{\partial z}-\frac{\partial u}{\partial x}\right)\hat{k}$$

Other vector relations

There are several other vector relations that may be helpful in fluid mechanics. These are shown below for a scalar $a$, and vectors $\vec{A}$ and $\vec{B}$.

$$\nabla \times (\nabla a) = 0$$

$$\nabla \cdot \left(\nabla \times \vec{A}\right) = 0$$

$$\nabla \cdot (a\vec{A}) = a\left(\nabla \cdot \vec{A}\right) + \left(\vec{A} \cdot \nabla\right) a$$

$$\nabla \times \left(a\vec{A}\right) = a\left(\nabla \times \vec{A}\right) + (\nabla a) \times \vec{A}$$

$$\left(\vec{A} \cdot \nabla\right)\vec{A} = \nabla\left(\frac{A^2}{2}\right) - \vec{A} \times \left(\nabla \times \vec{A}\right)$$

$$\nabla \times \left(\vec{A} \times \vec{B}\right) = \vec{A}\left(\nabla \cdot \vec{B}\right) + \left(\vec{B} \cdot \nabla\right)\vec{A} - \vec{B}\left(\nabla \cdot \vec{A}\right) - \left(\vec{A} \cdot \nabla\right)\vec{B}$$

$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}$$