

SYMMETRY GROUPS

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The quantification and classification of the symmetries that are found in nature are useful in obtaining a deeper understanding of structure, form, and interaction. This paper will discuss the fundamental principles that define isometries and the creation of symmetry groups. In particular, Frieze patterns will be analyzed to create a basis from which the properties of symmetry can be extended out to more complicated designs.

1 Introduction

Symmetry exists everywhere in nature: the repeated petals of a flower, the dendritic pattern of the veins in a leaf, the atomic arrangement in a crystal lattice, or the mirrored ridges of a scallop shell. A spinning rim on a car or a repeating molding around the top of a room also contains symmetries. In the quest to understand and imitate nature, it would be helpful to be able to quantify and classify symmetries that exist in a given design. Understanding the fundamental principles that govern the relationship among points in a design can assist in a deeper knowledge of a structural or aesthetic reason for the parallelism.

Cave paintings have been found to exhibit simple repeating, translational symmetry. The Greeks used this same sort of symmetry to design the patterns that wrapped around the tops of their temples. Cathedrals with their expansive naves and large rose windows are likewise filled with symmetries. Even more recent works of art such as those by M.C. Escher exploit principles of symmetry to create beauty.

2 Groups

Let us look at the set of integers beginning with a couple trivial facts.

1. The result of adding together two integers is another integer.
2. The result of adding an integer and 0 is that integer.
3. For any integer, there exists another integer such that when added together, the result is 0.
4. Suppose there are three integers a , b , and c . $a + b + c = a + (b + c)$.

Since the set of integers has those four properties, the set of integers under addition is a **group**.

Definition 0.1 *Let G be a nonempty set together with a binary operation. G is a group if*

1. The set is closed: Given two members of G , a and b , the binary operation assigns the order pair (a, b) another element in G .
2. Associativity: $(ab)c = a(bc)$
3. Identity: There exists an element such that $ae = ea = a$.
4. Inverses: For each element in G , there is an element b in G such that $ab = ba = e$.

3 Isometries

There are many ways to change a picture. Below are a number of possible ways to change the original image as found on the left.



Figure 0.1: Transformations

However, there is a fundamental transformation where the distance between any two points is preserved. This particular type of transformation is called an **isometry**.

Definition 0.2 An **isometry** of n -dimensional space R_n is a function from R_n onto R_n that preserves distance.

Let us look at R_1 . A set of points in R_1 is on a line. There are a number of isometries for a set of points on a line. There can be a translation. All of the points move along the line by a certain unit. There can also be reflections around some point. If the points don't move, we have the identity. We can prove that there are only three types of isometries in R_1 .

Proof: Prove that there are only three type of isometry in R_1 . Suppose we have an isometry. This isometry can either leave no point fixed, it can leave one point fixed, or it can leave two or more points fixed in R_1 . Note: $d(x, y)$ means the distance between x and y .

1. Suppose the isometry f leaves two or more point fixed. If f leaves two or more points fixed, there exists two points a, b such that $f(a) = a$, and $f(b) = b$. Suppose, I have an arbitrary point, x . By the definition of an isometry, $d(f(x), f(a)) = d(x, a)$. Since $f(a) = a$, $d(f(x), a) = d(x, a)$. Since the points are on a line, $d(x, a) = |x - a| = |f(x) - a|$. Similarly, by the same logic, $d(x, b) = |x - b| = |f(x) - b|$. If we square both expressions, we get:

$$\begin{aligned} (x - a)^2 &= (f(x) - a)^2 \\ (x - b)^2 &= (f(x) - b)^2 \end{aligned}$$

If we subtract the expressions, we get:

$$\begin{aligned} 2bx - 2ax &= 2bf(x) - 2af(x) \\ 2(b - a)x &= 2(b - a)f(x) \end{aligned}$$

Since both sides of the expression must be equal, $f(x) = x$. This means that if an isometry leaves two or more points fixed, any arbitrary point will map onto itself under this isometry. Thus, this isometry is the identity.

2. Suppose the isometry f leaves one point fixed. If f leaves one point fixed, there exists a point such that $f(a) = a$. Suppose, I have an arbitrary point, $a + x$. By the definition of isometry, $d(f(a + x), f(a)) = |a + x - a| = x$. Therefore, $f(a + x)$ is either $a + x$ or $a - x$. Since f only leaves one point fixed, $f(a + x) = a - x$. Therefore, the isometry fixed the point a , but $a + x$ is always mapped to $a - x$ for any arbitrary x . This is a reflection fixed at point a . Thus, if the isometry f leaves one point fixed, the isometry is a reflection.
3. Suppose the isometry f leaves no point fixed. If f leaves no points fixed there exists a point such that $f(x) = y$, where $y \neq x$. Either $y = a - x$, or y is some other arbitrary point. If $y = a - x$, this would imply f is a reflection (see case 2). Therefore, y must be some other point. This is a translation. Thus, if the isometry f leaves no point fixed, it is a translation.

Hence, there are only three type of isometry in R_1 . ■

Considering R_2 , similar methods can be employed to prove that in R_2 , there are five types of isometries: **identity, translation, rotation, reflection and glide-reflection.**

4 Symmetry Groups

Suppose there is a set of all possible isometries in R_n that carries a set of points back onto itself. Is the set a group under function composition?

Proof: The set of all possible isometries in R_n that carries a set of points back onto itself is a group. By the definition of a group, we have to show that the set has an identity, there exists an inverse for any member in the set, the operation is associative, and the set is closed.

1. There is an identity. For all x_1, x_2 in R_n , the identity is an isometry which maps x_1 and x_2 onto x_1 and x_2 , respectively. Since the set has all possible isometries in R_n that maps a set of points back onto itself, the identity must be a member of the set.
2. The composition of two isometries will be another isometry in the set. Suppose the set contains f and g , two isometries in R_n . For all x_1, x_2 in R_n , $d(f(x_1), f(x_2)) = d(g(x_1), g(x_2)) = d(x_1, x_2)$ by the definition of an isometry. This means for $f \circ g$, $d(f \circ g(x_1), f \circ g(x_2)) = d(x_1, x_2)$. Therefore $f \circ g$ is an isometry. Since f maps the set of points back onto itself, and g maps the set of points back onto itself, $f \circ g$ must also map the set of points back onto itself. Therefore, the composition of two isometries will be another isometry in the set.
3. For an isometry f , there exists an inverse, f^{-1} . Suppose the set contains f . $d(f(x_1), f(x_2)) = d(x_1, x_2)$. If f^{-1} exists for f , it will map the points back to original points, Since $d(x_1, x_2)$ have not changed, $d(f^{-1}(x_1), f^{-1}(x_2)) = d(x_1, x_2)$. Thus, f^{-1} is an isometry. Since f maps the set of points back onto itself, and f^{-1} undoes that transformation, f^{-1} must also be transformation that map points back onto itself. Thus f^{-1} exists in our set.
4. Prove $f \circ g \circ h = f \circ (g \circ h)$ Function composition is associative: Since $(f \circ g \circ h)(x) = f \circ g \circ (h(x)) = f(g(h(x)))$ and $f \circ (g \circ h)(x) = f((g \circ h)(x)) = f(g(h(x)))$, $f \circ g \circ h = f \circ (g \circ h)$

The set of all possible isometries in R_n that carries a set of points back onto itself is a group under function composition. ■

Definition 0.3 *Symmetry Group of a Figure in R^n : Let F be a set of points in R^n . The symmetry group of F in R^n is the set of all isometries of R^n that carry F onto itself. The group operation is function composition.*

Here is an example of a finite group. Below is a Nanu flower, an endangered plant from Hawaii:



Figure 0.2: Nanu flower, 6-fold rotational symmetry

There is a corresponding symmetry group for the flower. Looking at this flower with six petals, there are two types of isometries that will carry the flower back onto itself. There are a number of rotations around the center of the flower, and there are also a number of reflections about axes of symmetry that intersect the center of the flower. There are six unique rotations: 0° , 60° , 120° , 180° , 240° , or 300° . There are also six different lines of reflectional symmetry. Each reflection line cuts through the center of a different petal, or the space between two petals. Since there are no other isometries that will map the flower back onto itself, the set of these isometries is a symmetry group.

5 Frieze Groups

Frieze Groups Frieze patterns are 1-dimensionally tessellating designs of a finite unit length. To create a Frieze pattern, a unit-length design is repeatedly translated linearly by a fixed distance in opposite directions such that the resulting strip of design is periodic and infinite in length.

Frieze patterns can be classified by the symmetry group of the unit-length design that is the building block of a pattern. Vertically, there is only one type of symmetry that exists: reflectional symmetry. Therefore a design may have two possible vertical classifications: reflectional (labeled m) or no symmetry (labeled 1). Horizontally, there are three types of symmetry that exist: reflectional (labeled m), glide (labeled g), and 2-fold rotational symmetry (labeled 2). Additionally, a pattern may not contain any horizontal symmetries (labeled 1), resulting in a total of four possible horizontal classifications. Therefore, by the product rule, given the possible two vertical and four horizontal symmetries, there are a total of eight permutations of vertical and horizontal symmetries, creating symmetry groups of Frieze patterns.

However, despite the fact that there should be eight symmetry groups, all Frieze patterns can be described by only seven of them. The $m2$ group is typically not considered one of the seven fundamental symmetry groups because it can be represented by either of two other groups. If the suspected 2-fold roto-center lies on the vertical reflectional axis, the pattern is an mm , and if it is located in any other location, the pattern is an mg .

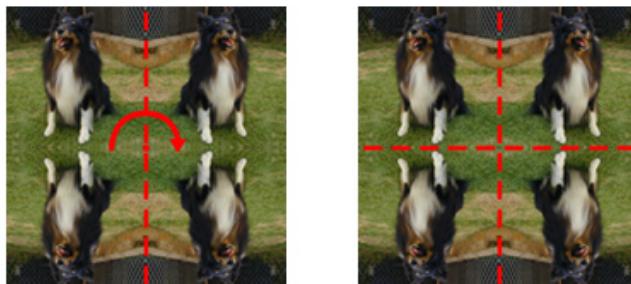


Figure 0.3: $m2$ (left image) is actually an mm (right image)



Figure 0.4: $m2$ (left image) is actually an mg (right image)

These Frieze pattern symmetry groups represent all possible isometries of the original image because we have generated them from the fundamental component symmetries that exist in a unit-length pattern. Therefore these are the only isometries which map the image back on to itself.

6 Conclusion

While frieze groups represent some of the simplest symmetry groups, the same principles that govern the creation of such groups of isometries can be extended out to greater dimensions of space. Two dimensional symmetry groups are often referred to as crystallographic or wallpaper groups, referencing their common use as patterns on decorative wall coverings. Three dimensional symmetry groups are particularly important to biological, chemical, and physical systems as they describe the packing of atoms, molecules, or structures in the world in which we live. The assortment of petals around a flower or the cubic structure of the components of table salt are all intentional patterns related back to chemical or structural reasons.

The ability to understand the symmetries found in nature is the first step to being able to imitate them. Links to symmetries are being found today that extend all the way down to the molecular level of matter. Atoms in crystals as well as the honeycomb of bees can be found in patterns of tessellated hexagons. Enantiomers have been found in biology which are isometries of molecules. They are mirror-image molecules which are composed of the same atoms but shaped in an opposite ways.

Even further, the age old question, "What is beauty?" can partially be answered by symmetry. Symmetrical patterns, while so prevalent in our daily lives, are rarely quantified or understood. For the majority of people it is sufficient to say that something looks nice or is beautiful without understanding why. Symmetry can be related to harmony or balance and proportionality, a visual embodiment of the yin-yang relationship. While it is still not fully understood, studies have been performed which relate a higher degree of feature symmetry in natural beings to higher perceived attractiveness. Perhaps the presence of these properties of symmetry lead a mind to feel more comfortable or at ease when being viewed. Symmetry has more to teach us than we currently see.